1. A Simple Circuit

For the circuit shown below, find the voltages across all the elements and the currents through all the elements.

(a) In the above circuit, pick a ground node. Does your choice of ground matter?
   **Answer:**
   There are two nodes in this circuit and thus two choices for the ground node. The choice of ground does not matter. We will use the ground node shown below:

   ![Electrical Circuit Diagram](image)

(b) With your choice of ground, label the node potentials for every node in the circuit.
   **Answer:**
   Since this circuit only has two nodes, there will only be one additional node potential.

   ![Electrical Circuit Diagram](image)

(c) Label all of the branch currents. Does the direction you pick matter?
   **Answer:**
   When labeling the currents through branches, the direction you pick does not matter.

   ![Electrical Circuit Diagram](image)
(d) Draw the $+/−$ labels on every element. What convention must you follow?

**Answer:**

When drawing the $+/−$ labels, you must follow the passive sign convention. That is, current flows into the $+$ terminal of every element.

![Diagram of a circuit with labels](image)

(e) Set up a matrix equation in the form $A\vec{x} = \vec{b}$ to solve for the unknown node potentials and currents. What are the dimensions of the matrix $A$?

**Answer:**

$$ \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} $$

$A$ will be a $3 \times 3$ matrix since there are three unknowns in the circuit, the two currents $I_0$ and $I_1$ and the one potential $u_1$.

(f) Use KCL to find as many equations as you can for the matrix.

**Answer:**

KCL gives us one equation for the node at the top, namely that $I_0 - I_1 = 0$. Thus, so far our matrix is as follows:

$$ \begin{bmatrix} 1 & -1 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ ? \\ ? \end{bmatrix} $$

(g) Use $IV$ relations to find the remaining the equations for the matrix.

**Answer:**

We know that the difference in potentials across the voltage source must be the voltage on the voltage source, i.e.

$$ -V_0 = V_S \quad (1) $$

We also know that the voltage across the resistor is equal to the current times the resistance, i.e.

$$ V_1 = I_1R_1 \quad (2) $$

Writing the equations for node potentials we have:

$$ 0 - u_1 = V_0 $$
$$ u_1 - 0 = V_1 $$

Substituting expressions from Equations (1) and (2) into Equation (3), we have:

$$ -u_1 = -V_S \implies u_1 = V_S $$
$$ u_1 = I_1R_1 \implies -I_1R_1 + u_1 = 0 $$

(4)
Our matrix is then:
\[
\begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 1 \\
0 & -R_1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
u_1 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
V_S \\
0 \\
\end{bmatrix}
\]

(h) Solve the system of equations if \(V_S = 5\) V and \(R_1 = 5\) \(\Omega\).

**Answer:**
By plugging the given values into the system of equations, we get:
\[
\begin{bmatrix}
I_0 \\
I_1 \\
u_1 \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
5 \\
\end{bmatrix}
\]

2. A Slightly More Complicated Circuit

For the circuit shown below, find the voltages across all the elements and the currents through all the elements.

(a) In the above circuit, pick a ground node. Does your choice of ground matter?

**Answer:**
There are two nodes in this circuit and thus two choices for the ground node. The choice of ground does not matter. We will use the ground node shown below:

(b) With your choice of ground, label the node potentials for every node in the circuit.

**Answer:**
Since this circuit only has two nodes, there will only be one additional node potential.
(c) Label all of the branch currents. Does the direction you pick matter?

**Answer:**
When labeling the currents through branches, the direction you pick does not matter.

![Branch Circuit Diagram]

(d) Draw the +/- labels on every element. What convention must you follow?

**Answer:**
When drawing the +/- labels, you must follow the passive sign convention. That is, current flows into the + terminal of every element.

![Sign Convention Diagram]

(e) Set up a matrix equation in the form \( A\vec{x} = \vec{b} \) to solve for the unknown node potentials and currents. What are the dimensions of the matrix \( A \)?

**Answer:**
\[
\begin{bmatrix}
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
? \\
? \\
? \\
? \\
\end{bmatrix}
\]

\( A \) will be a 4 \times 4 matrix since there are four unknowns in the circuit, the currents \( I_0, I_1, \) and \( I_2 \) and the one potential \( u_1 \).

(f) Use KCL to find as many equations as you can for the matrix.

**Answer:**
KCL gives us one equation for the node at the top, namely that \( I_0 - I_1 - I_2 = 0 \). Thus, so far our matrix is as follows:
\[
\begin{bmatrix}
1 & -1 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
? \\
? \\
? \\
\end{bmatrix}
\]

(g) Use IV relations to find the remaining the equations for the matrix.

**Answer:** We know that the current through the current source must be the value of the current source, i.e.
\[
I_0 = I_S
\]
We also know that the voltage across the resistor is equal to the current times the resistance, i.e.

\[
V_1 = I_1R_1 \\
V_2 = I_2R_2
\]  

(6)

Writing the equations for node potentials we have:

\[
0 - u_1 = V_0 \\
u_1 - 0 = V_1 \\
u_1 - 0 = V_2
\]  

(7)

Using Equation (5) and substituting expressions from Equation (6) into Equation (7), we have:

\[
I_0 = I_S \\
u_1 = I_1R_1 \implies -I_1R_1 + u_1 = 0 \\
u_1 = I_2R_2 \implies -I_2R_2 + u_1 = 0
\]  

(8)

Our matrix is then:

\[
\begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & 0 \\
0 & -R_1 & 1 \\
0 & 0 & -R_2 \\
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
I_S \\
0 \\
0 \\
\end{bmatrix}
\]  

(h) Solve the system of equations if \(I_S = 5 \text{ A}, R_1 = 5 \Omega, \text{ and } R_2 = 10 \Omega\).

**Answer:**

By plugging in the values into the system of equations, we get:

\[
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1 \\
\end{bmatrix}
= \begin{bmatrix}
5 \\
3.33 \\
1.67 \\
16.67 \\
\end{bmatrix}
\]

3. **(PRACTICE) Another Circuit**

For the circuit shown below, find the voltages across all the elements and the currents through all the elements.

\[
\begin{bmatrix}
V_5 \\
\end{bmatrix}
\begin{bmatrix}
+ \\
R_1 \\
R_2 \\
\end{bmatrix}
\]

(a) In the above circuit, pick a ground node. Does your choice of ground matter?

**Answer:**

There are two nodes in this circuit and thus two choices for the ground node. The choice of ground does not matter. We will use the ground node shown below:
(b) With your choice of ground, label the node potentials for every node in the circuit.

**Answer:**

Since this circuit only has two nodes, there will only be one additional node potential.

(c) Label all the branch currents. Does the direction you pick matter?

**Answer:**

When labeling the currents through branches, the direction you pick does not matter.

(d) Draw the +/- labels on every element. What convention must you follow?

**Answer:**

When drawing the +/- labels, you must follow the passive sign convention. That is, current flows into the + terminal of every element.

(e) Set up a matrix equation in the form $A\vec{x} = \vec{b}$ to solve for the unknown node potentials and currents. What are the dimensions of the matrix $A$?

**Answer:**

$$
\begin{bmatrix}
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
? \\
? \\
? \\
? \\
\end{bmatrix}
$$
A will be a $4 \times 4$ matrix since there are four unknowns in the circuit, the currents $I_0$, $I_1$, and $I_2$ and the one potential $u_1$.

(f) Use KCL to find as many equations as you can for the matrix.

Answer:
KCL gives us one equation for the node at the top, namely that $I_0 - I_1 - I_2 = 0$. Thus, so far our matrix is as follows:

$$
\begin{bmatrix}
1 & -1 & -1 & 0 \\
? & ? & ? & ?
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
? \\
? \\
?
\end{bmatrix}
$$

(g) Use IV relations to find the remaining equations for the matrix.

Answer: We know that the difference in potentials across the voltage source must be the voltage on the voltage source, i.e.

$$-V_0 = V_S. \tag{9}$$

We also know that the voltage across the resistor is equal to the current times the resistance, i.e.

$$V_1 = I_1 R_1 \tag{10}$$
$$V_2 = I_2 R_2$$

Writing the equations for node potentials we have:

$$0 - u_1 = V_0 \tag{11}$$
$$u_1 - 0 = V_1$$
$$u_1 - 0 = V_2$$

Substituting expressions from Equations (9) and (10) into Equation (11), we have:

$$-u_1 = -V_S \implies u_1 = V_S$$
$$u_1 = I_1 R_1 \implies -I_1 R_1 + u_1 = 0$$
$$u_1 = I_2 R_2 \implies -I_2 R_2 + u_2 = 0$$

Our matrix is then:

$$
\begin{bmatrix}
1 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & -R_1 & 0 & 1 \\
0 & 0 & -R_2 & 1
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
V_S \\
0 \\
0
\end{bmatrix}
$$

(h) Solve the system of equations if $V_S = 5\, \text{V}$, $R_1 = 5\, \Omega$, and $R_2 = 10\, \Omega$.

Answer:
By plugging in the values into the system of equations, we get:

$$\begin{bmatrix}
I_0 \\
I_1 \\
I_2 \\
u_1
\end{bmatrix}
= 
\begin{bmatrix}
1.5 \\
1 \\
0.5 \\
5
\end{bmatrix}$$
4. Mechanical Circuits

Find the voltages across and currents flowing through all of the resistors.

Answer:
First, we label the ground node. Then, we label all the node potentials, branch currents and identify +/- labels for each element:

Expressing the voltage differences in terms of node potentials, we get

\[ V_0 = 0 - u_1 \]
\[ V_1 = u_1 - u_2 \]
\[ V_2 = u_2 - 0 \]
\[ V_3 = u_2 - u_3 \]
\[ V_4 = u_3 - 0 \]

Now we set up our KCL equation:

\[ I_1 - I_2 - I_{34} = 0 \]
We can use KVL and $IV$ relations to find the rest of the equations. We know that the difference in potentials across the voltage source must be the voltage on the voltage source, i.e.

$$-V_0 = 10.$$  \hfill (14)

We also know that the voltage across the resistor is equal to the current times the resistance, i.e.

$$V_0 = -10$$

$$V_1 = I_1 R_1$$

$$V_2 = I_2 R_2$$

$$V_3 = I_{34} R_3$$

$$V_4 = I_{34} R_4$$  \hfill (15)

Substituting expressions from Equations (14) and (15) into Equation (13), we have

$$u_1 = 10$$

$$u_1 - u_2 = I_1 R_1 \implies -I_1 R_1 + u_1 - u_2 = 0$$

$$u_2 = I_2 R_2 \implies -I_2 R_2 + u_2 = 0$$

$$u_2 - u_3 = I_{34} R_3 \implies -I_{34} R_3 + u_2 - u_3 = 0$$

$$u_3 = I_{34} R_4 \implies -I_{34} R_4 + u_3 = 0$$

We can now set the system up in a matrix-vector product form and use Gaussian Elimination/IPython to solve:

$$\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-R_1 & 0 & 0 & 1 & -1 & 0 \\
0 & -R_2 & 0 & 0 & 1 & 0 \\
0 & 0 & -R_3 & 0 & 1 & -1 \\
0 & 0 & -R_4 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_{34} \\
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
10 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.$$  \hfill (13)

This returns the array:

$$\begin{bmatrix}
I_1 \\
I_2 \\
I_{34} \\
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix}
2 \\
\frac{4}{3} \\
\frac{2}{3} \\
10 \\
4 \\
3
\end{bmatrix}.$$  \hfill (13)

Substituting the values of node potentials in Equation (13), we have

$I_1 = 2$ A,

$I_2 = 4/3$ A,

$I_{34} = 2/3$ A,

$V_1 = 6$ V,

$V_2 = 4$ V,

$V_3 = 3$ V,

$V_4 = 1$ V.