EE16A
Sept 11, 2018.
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OH: Today 144MA
Thu: 212 Cory.
Website updated.

Reminders: ① HW2 Self grades - today.
② HW3 Friday midnight.
↳ Self problem.
③ EECS Colloquium: Ranveer Chandra
Farm Beats: Empowering Farmers with AI solutions for Agriculture.
Wed 4-5 pm 306 Soda.

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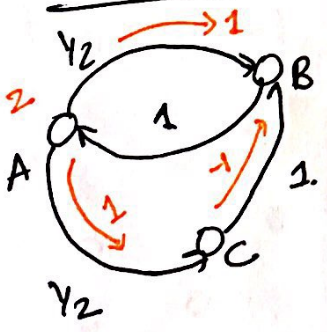
Today:
- More inversion — Gauss - Jordan Method
  ↳ Going to Identity.
- Matrix multiplication as transformation of spaces.
- Vector space
- Null space.

Last time



$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x}(t-1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{x}(t+1) = Q\,\vec{x}(t)}$$

Write $\vec{x}(t-1)$ as a function of $\vec{x}(t)$

$$\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = Q \cdot \vec{x}(t-1) = Q \cdot P \cdot \vec{x}(t)$$

$$\vec{x}(t) = Q\,\vec{x}(t-1)$$

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

$$\boxed{Q \cdot P = I}$$

$$\vec{x}(t-1)$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Def: Inv. of Q as a matrix P such that
$$P \cdot Q = Q \cdot P = I.$$

Q. P. = I Find P.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3(-1)} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

Aside

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

$$A\vec{x}_1 = \vec{b}_1$$
$$A\vec{x}_2 = \vec{b}_2$$
$$A\begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix}$$

Gauss-Jordan Elimination.
↳ 1888

Clausen 1888.

$Q \cdot P = I.$

Thm: Left inverse = Right inverse

Proof: ~~Say~~ $Q \cdot P = I$

P right mult. Q.

Q left mult P

Let $R \cdot Q = I$

$$(R \cdot Q) \cdot P = I \cdot P$$
$$R(QP) = P$$
$$R \cdot I = P \Rightarrow R = P.$$

If $Q \cdot P = I$

then, $P \cdot Q = I$.

Thm: Matrix inverse is unique.

Proof: If P_1 s.t. $P_1 Q = Q \cdot P_1 = I$.

P_2 s.t. $P_2 Q = Q \cdot P_2 = I$.

$$P_2 = I \cdot P_2 = P_1 \cdot Q \cdot P_2 = P_1 \cdot I = P_1$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \qquad P = Q^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

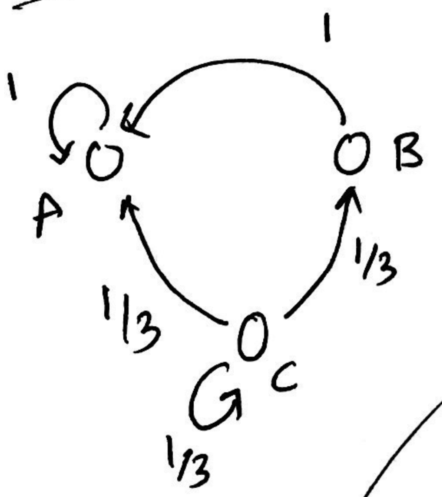
Q

$\underline{P \cdot Q = I}$ $\qquad$ $\underline{Q \cdot P \cdot = I}$

Do you always have an inverse?

→ NO. :~

$0 \qquad \frac{1}{0} \qquad \frac{1}{5}$



$$Q = \begin{bmatrix} 1 & 1 & 1/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}(t-1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Row 3 - Row 2 : Row of zeros!

Sometimes we have no solution.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1/3 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1 \end{array}\right]$$

$[0\ 0\ 0\ 0 \cdots 0 \mid b \neq 0] \rightarrow$ no solutions.

<u>Claim:</u> If A is an invertible matrix, then $A\vec{x} = \vec{0}$ has only one unique solution, $\vec{x} = \vec{0}$

Homogeous eqn.

$A\vec{x} = \vec{b}$

Let A^{-1} be the inverse of A

$A^{-1}A \cdot \vec{x} = A^{-1} \cdot \vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$x = 5y$

$y = 1/5\, x = (5^{-1})x.$

Let us say. $\vec{x_1}$ s.t.

$$\begin{bmatrix} | & | & | \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\vec{x_1} = \vec{0}$

$A^{-1} \cdot A \cdot \vec{x_1} = A^{-1} \cdot \vec{0}$

$I \cdot \vec{x_1} = \vec{0}$

$\Rightarrow \vec{x_1} = \vec{0}$

Thm: Matrix A is invertible if and only if the columns of A are linearly independent.

Part 1: If A^{-1} exists,

then, $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$.

$\Rightarrow$ Cols of A are linearly independent.

Part 2: If columns of A ar lin. indep.

$A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$.

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \checkmark$$

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$\Rightarrow$ Inverse exists!