EE16A
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OH Today 11-12noon
9/25/2018

Reminders

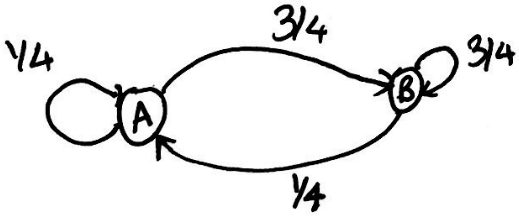
MT1: Monday, Oct 1.
HW due Friday
Self-grades today.

Today: "Connecting up"

- Nullspaces $\leftrightarrow$ eigenvalues.
- Eigenbasis
- Diagonalization.

Definition: Eigenvector: Matrix A
vector $\vec{v} \neq 0$ such that $A\vec{v} = \lambda \vec{v}$

Connections: Nullspaces $\leftrightarrow$ Eigenvalues.



$$Q = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}.$$

$$Q - \lambda I = \begin{bmatrix} 1/4 - \lambda & 1/4 \\ 3/4 & 3/4 - \lambda \end{bmatrix}$$

$$\det(Q - \lambda I) = \left(\frac{1}{4} - \lambda\right)\left(\frac{3}{4} - \lambda\right) - \frac{3}{16}$$

$$= \lambda^2 - \lambda = \lambda(\lambda - 1).$$

$$\text{Eigenvalues}(Q) = \{\lambda = 0, \lambda = 1\}.$$

$$\text{Null}\left(\begin{bmatrix} 1/4 - 1 & 1/4 \\ 3/4 & 3/4 - 1 \end{bmatrix}\right)$$

$$\rightarrow \begin{bmatrix} -3/4 & 1/4 \\ 3/4 & -1/4 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t.$

$-3x_1 + x_2 = 0 \qquad -3x_1 + t = 0 \implies 3x_1 = t$

e-space : $\text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$. Corresponding to $\lambda = 1$.

e-space to $\lambda = 0$.

$(Q - \lambda I) = Q$

$$\text{Null}(Q) = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1/4 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$\boxed{\text{Null}(Q) : \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$

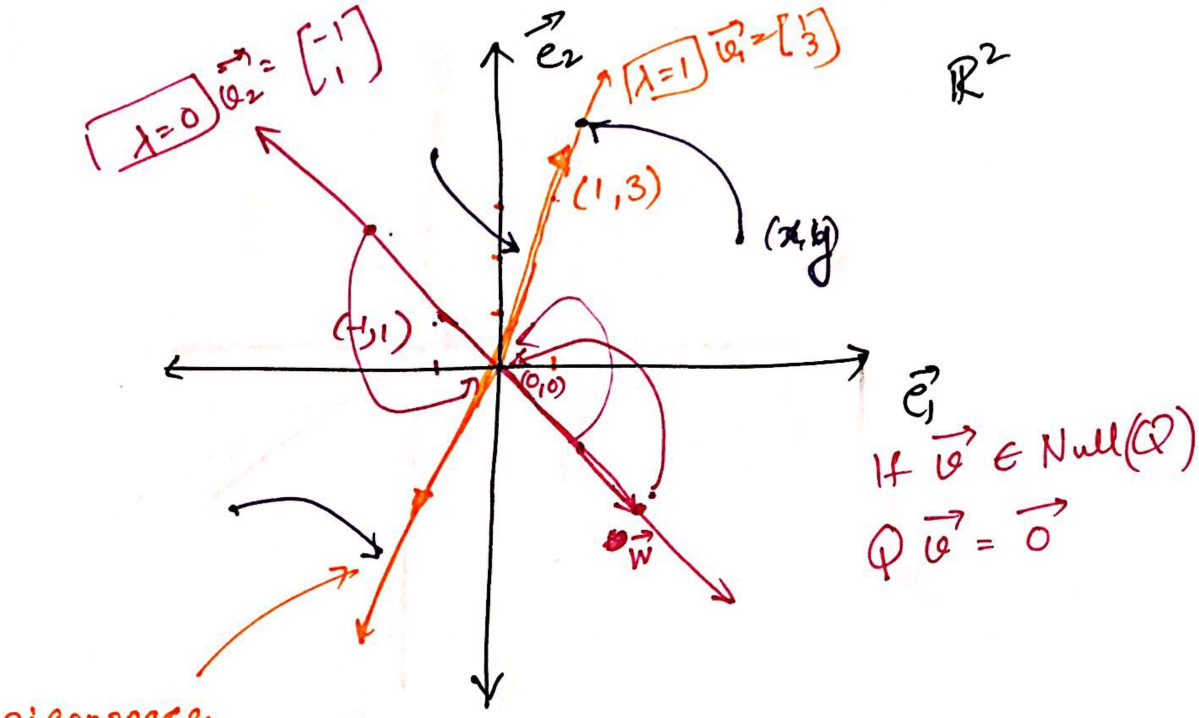
Eigenspace corresponding to $\lambda = 0$, is the nullspace (Q)

Columnspace:

$$\boxed{Q\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} \frac{1}{4}(x+y) \\ \frac{3}{4}(x+y) \end{bmatrix} = \left(\frac{x+y}{4}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

$\mathbb{R}^2$



eigenspace corresponding to $\lambda = 1$

$Q \cdot \vec{v} = 1 \cdot \vec{v}$

Q : non-invertible transform.

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad Q \cdot \vec{v_1} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ $\quad \lambda_1 = 3.$

$\lambda_2 = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 + x_3 = 0$

$x_2 = t_2$

$x_3 = t_3$

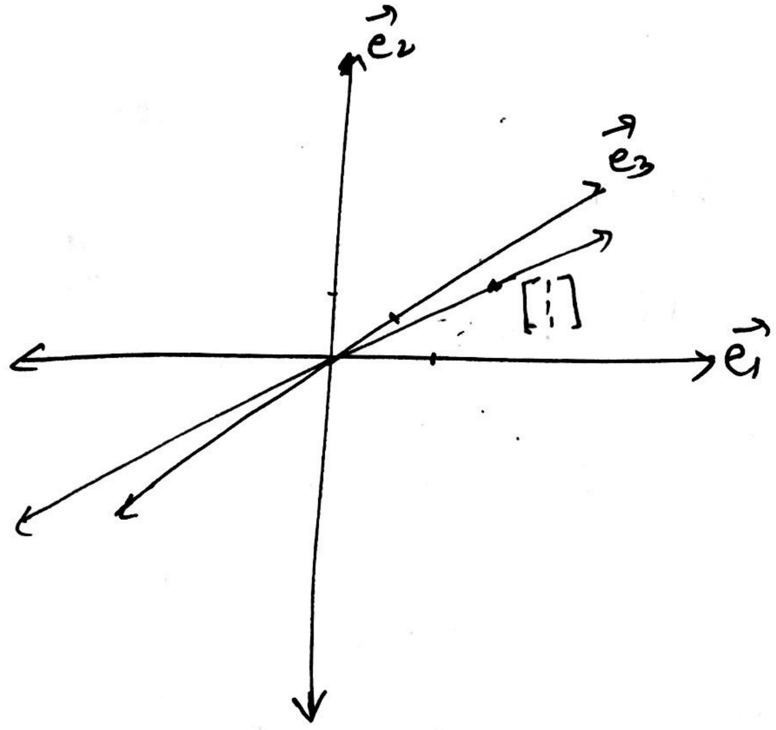
$x_1 = -t_2 - t_3$

$\text{span}\left\{ \begin{bmatrix} -t_2 - t_3 \\ t_2 \\ t_3 \end{bmatrix} \right\}$

$t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\text{Null}(Q)$

$= \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$



Nilpotent matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ $\lambda = 0$ as a repeated e-value.

$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Diagonalization :

Last lecture: If $\lambda_1, \lambda_2 \cdots \lambda_n$ are distinct eigenvalues, of matrix Q then $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ form a basis for $\mathbb{R}^n$.

$\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n \rightarrow$ "eigenbasis".

Standard Basis $\qquad$ Eigenbasis.

$$\vec{x} \longleftrightarrow \vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots \alpha_n \vec{v}_n = V \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

↳ co-ordinates.

$$\vec{x} = V \cdot \vec{x}_V \qquad \vec{x}_V = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\vec{x}_V = V^{-1} \cdot \vec{x}$$

$$\boxed{Q \cdot \vec{v} = \lambda \vec{v}}$$

$$Q \cdot \vec{x} = Q(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n)$$

$$= \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 + \cdots + \alpha_n \lambda_n \vec{v}_n$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{V} \begin{bmatrix} \alpha_1 \lambda_1 \\ \alpha_2 \lambda_2 \\ \vdots \\ \alpha_n \lambda_n \end{bmatrix}$$

$$= V \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = V \Lambda V^{-1} \cdot \vec{x}$$

$$Q \cdot \vec{x} = V \cdot \Lambda \cdot V^{-1} \cdot \vec{x} \rightarrow \text{Diagonal } \Lambda \qquad \vec{x}_V = V^{-1} \vec{x}$$

$$\boxed{Q = V \Lambda V^{-1}}$$

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$$

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$$Q = V \Lambda V^{-1}$$

$$Q^2 = (V \Lambda V^{-1})(V \Lambda V^{-1})$$

$$= (V \Lambda I \Lambda V^{-1})$$

$$= V \Lambda^2 V^{-1}$$

$$Q^m = V \Lambda^m V^{-1}$$ ← Diagonal form.

Λ is the representation of Q in eigenbasis.

$$A \cdot \vec{v}_1 = \frac{1}{2} \vec{v}_1$$

$$A \cdot \vec{v}_2 = \frac{1}{3} \vec{v}_2$$

$$\Lambda = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$