EE16A

Nov 15, 2018

1

- OH today after class.
- HW 12 due tomorrow.
- Extra OH tomorrow 10-11 am.
  - ↳ Catch-up.

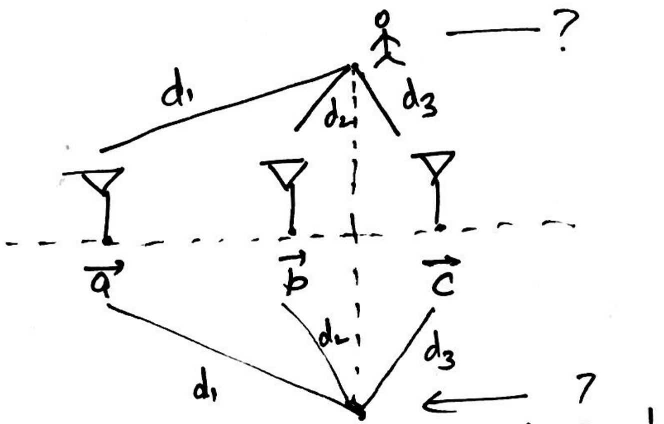
Today:

- Least Squares Revisited
- Importance of measurement matrix
- How to use least squares.

---

Trilateration:

$$\underbrace{2\begin{bmatrix} (\vec{a}-\vec{b})^T \\ (\vec{a}-\vec{c})^T \end{bmatrix}}_{A} \vec{x} = \begin{bmatrix} d_2^2 - d_1^2 + \|\vec{a}\|^2 - \|\vec{b}\|^2 \\ d_3^2 - d_1^2 + \|\vec{a}\|^2 - \|\vec{c}\|^2 \end{bmatrix}$$




Doesn't work if your beacons are collinear!

---

Least Squares - Overdetermined system.

A : not square.

$$\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \underset{\text{unknown}}{\begin{bmatrix} \vec{x} \end{bmatrix}} \approx \begin{bmatrix} \vec{b} \end{bmatrix}$$

$m > n$ $\quad A_{m \times n} \quad \vec{x}_{n \times 1} \quad$ data $\vec{b} \to m \times 1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{E} \end{array}\right]$$

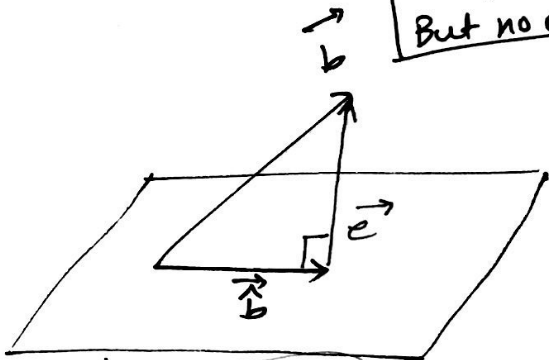
$\hookrightarrow$ no solution.

If $\vec{b} \in \text{span}(A) \rightarrow$ ~~[illegible]~~ solution.

Want to solve $A\vec{x} = \vec{b}$ But no exact $\text{sol}^n$

Case of interest: $\vec{b} \notin \text{span}(A)$.

$$A\vec{x} = \vec{\hat{b}} + \vec{e} = \vec{b}$$




Want: Minimize $\|\vec{e}\|^2$ error

Want: $\vec{b} - \vec{\hat{b}} = \vec{e} \perp \text{span}(A)$

$A\hat{\vec{x}} = \vec{\hat{b}}$

estimate of $\vec{x}$ (true)

approximation for $\vec{b}$ in the span(A)

Orthogonal to every vector in the span(A).

$\langle \vec{b} - \vec{\hat{b}}, \vec{a_1} \rangle = 0 \Rightarrow (\vec{b} - \vec{\hat{b}})^T \vec{a_1} = 0$

$\Rightarrow \vec{a_1}^T (\vec{b} - \vec{\hat{b}}) = 0$.

$\langle \vec{b} - \vec{\hat{b}}, \vec{a_2} \rangle = 0$

$\vdots$

$\langle \vec{b} - \vec{\hat{b}}, \vec{a_n} \rangle = 0$

$$\vec{a}_1^T \cdot \vec{b} = \vec{a}_1^T \cdot \hat{\vec{b}}$$

$$\vdots$$

$$\vec{a}_n^T \cdot \vec{b} = \vec{a}_n^T \cdot \hat{\vec{b}}$$

$$\rightarrow \underbrace{\begin{bmatrix} - \vec{a}_1^T - \\ \vdots \\ - \vec{a}_n^T - \end{bmatrix}}_{A^T} \vec{b} = \underbrace{\begin{bmatrix} - \vec{a}_1^T - \\ \vdots \\ - \vec{a}_n^T - \end{bmatrix}}_{A^T} \hat{\vec{b}}$$

$$A^T \vec{b} = A^T \hat{\vec{b}} = A^T \cdot A \cdot \hat{\vec{x}}$$

$$\underset{n\times m}{A^T} \ \underset{m\times n}{A} \rightarrow \boxed{A^T A_{n\times n}}$$

$$A^T \vec{b} = A^T A \cdot \hat{\vec{x}}$$

$$\boxed{\begin{aligned} \hat{\vec{x}} &= (A^T A)^{-1} \cdot A^T \cdot \vec{b} \quad \text{(estimate of } \vec{x}\text{)} \\ \hat{\vec{b}} &= A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{b} \quad \text{(estimate/projection of } \vec{b} \text{ on span}(A)\text{)} \end{aligned}}$$

$\hookrightarrow$ <u>if $A^T A$ is invertible</u>

---

<u>Investigation:</u> Invertibility of $A^T A \in \mathbb{R}^{n\times n}$

<u>Connection:</u> Invertibility $\leftrightarrow$ Nullspaces.

$$\boxed{\text{Thm: } \quad \text{Null}(A^T A) = \text{Null}(A).}$$

If $A$ has a non-trivial Null($A$) $\Leftrightarrow$ columns of $A$ are linearly dependent.

<u>Proof:</u> (1) if $\vec{v} \in N(A)$ then $\vec{v} \in N(A^T A)$

(2) If $\vec{w} \in N(A^T A)$ then $\vec{w} \in N(A)$.

(1) $\vec{v} \in N(A)$.

$A \cdot \vec{v} = \vec{0} \longrightarrow (A^T A)\vec{v} = A^T \cdot \vec{0} = \vec{0}$

$\vec{v} \in N(A^T A)$

(2) $\vec{w} \in N(A^T A)$

$A^T A \cdot \vec{w} = 0 \longrightarrow A\vec{w} = 0.$

Multiply by $\vec{w}^T$

$(\vec{w}^T \cdot A^T)(A \cdot \vec{w}) = \vec{w}^T \cdot 0 = 0.$

$\boxed{\vec{x}^T \cdot \vec{x}}$

$= \|\vec{x}\|^2$

$\| A\vec{w} \|^2 = 0.$

$\Rightarrow A\vec{w} = 0.$

$\vec{w} \in N(A).$

Aside: Transpose . $\underset{m\times n}{(A} \cdot \underset{n\times m}{B)}^T = B^T A^T$ $A^T, B^T$ (5)

$A^T: n\times m \quad B^T: m\times n$

e.g. $\left(\underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\right)^T = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$ ✓

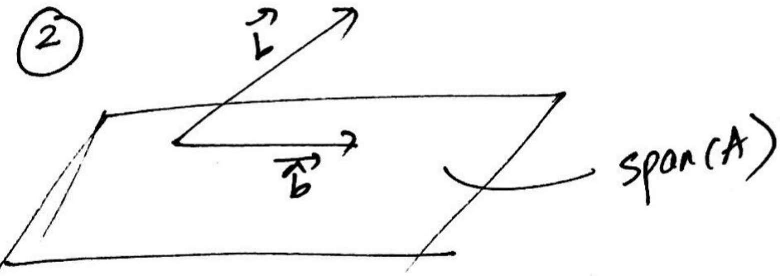
$\left(\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}^T.$

# Interpretations of LS

① Approximate solution to systems of linear equations.

$$A\vec{x} \approx \vec{b}$$

②




Projection interpretation.

Find $\vec{x}$ ← argument: $\underset{\vec{x}}{\text{argmin}} \; \| A\vec{x} - \vec{b} \|$

③ Machine-learning / prediction / regression / curve-fitting.

---

## Financial market

**Known:** $x_i, y_i$

**Unknown:** $\alpha, \beta$.

$$\underset{\text{price}}{y} = \underset{\text{slope}}{\alpha} \cdot \underset{\text{month}}{x} + \underset{\text{intercept}}{\beta}$$

Matrix-Vector eq. $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$y_1 = \alpha x_1 + \beta$$
$$y_2 = \alpha x_2 + \beta$$
$$\vdots$$
$$y_n = \alpha \cdot x_n + \beta.$$

$$\rightarrow \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Curve-fitting : Origin of least squares.

1800 — Gauss.

Ceres Piazzi

Kepler's Law: Elliptical orbits.

$(x_1, y_1)$, $(x_2, y_2)$ $\cdots$ $(x_n, y_n)$

Unknowns.

Ellipse:

$$\underline{a}x^2 + \underline{b}xy + \underline{c}y^2 + \underline{d}x + \underline{e}y = 1$$

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 \\ & & \vdots & & \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$