This homework is due Friday, September 20, 2019, at 23:59.
Self-grades are due Monday, September 23, 2019, at 23:59.

Submission Format
Your homework submission should consist of one file.

- `hw3.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.
  
  If you do not attach a PDF “printout” of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

- Practice problems will not be graded for homeworks, but are in scope for exams.

Submit each file to its respective assignment on Gradescope.

0. (PRACTICE) Matrix Multiplication

Learning Objective: Practice evaluating matrix-matrix multiplication.

(a) Given \( A \in \mathbb{R}^{3 \times 2} \) and \( B \in \mathbb{R}^{2 \times 4} \), what do you expect the dimensions of the matrix-matrix product \( AB \) to be?

(b) Compute the following matrix-matrix product:

\[
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & -1 & 0 \\
-3 & 0 & 2 & -1 \\
\end{bmatrix}.
\]

1. Figuring Out The Tips

Learning Objective: This problem showcases how you can understand general systems of equations by looking at simpler examples. In particular, see if you can generalize your intuition from the case of 5 and 6 guests to a general number of guests.

A number of guests gather around a round table for a dinner. Between every adjacent pair of guests, there is a plate for tips. When everyone has finished eating, each person places half their tip in the plate to their left and half in the plate to their right. Suppose you can only see the amount of tips in each plate after everyone has left. Can you deduce the amount that each individual tipped?

Note: For this question, if we assume that tips are positive, then we need to introduce additional constraint that would make the system of equations no longer linear. We are going to ignore this constraint and assume that negative tips are acceptable.
(a) Suppose six guests sit around a table and there are six plates of tips. If we know the amount of tip in each plate, $P_1$ to $P_6$, can we determine each individual’s tip amount, $G_1$ to $G_6$? If yes, explain why by examining the relationship between the plate values, $P_1$ to $P_6$, and guest tips, $G_1$ to $G_6$. If not, give two different assignments of $G_1$ to $G_6$ that will result in the same $P_1$ to $P_6$.

(b) Now let’s consider five guests at the table, $G_1$ to $G_5$, and we can see the amount of tips in the five plates, $P_1$ to $P_5$. In this new setting can you figure out each guest’s tip values, $G_1$ to $G_5$?

(c) If $n$ is the total number of guests sitting around a table, for which values of $n$ can you figure out everyone’s tip? You do not have to rigorously prove your answer. (Hint: consider what is different about parts a and b.)

2. Show It!

Learning Objectives: This is an opportunity to practice your proof development skills. For proofs you have seen before, use this as an opportunity to make sure you understood them by trying them independently using the steps discussed in class.

(a) Show that if the system of equations, $A\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $A$ are linearly dependent.

(b) (PRACTICE) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

(c) Let $n$ be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of $k$ linearly dependent vectors in $\mathbb{R}^n$. Show that for any $n \times n$ matrix $A$, the set $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ is a set of linearly dependent vectors.

3. Quadcopter Transformations

Learning Objectives: Linear algebra is often used to represent transformations in robotics. This problem introduces some of the basic uses of transformations.
Professor Boser and his colleagues are interested in testing a concept to establish a communication link to a quadcopter by laser. Consider a vector $\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ representing the location of the quadcopter relative to the origin. The quadcopter is only capable of three different maneuvers relative to the origin. The maneuvers are rotations about the x, y, and z axes. For perspective, the positive x-axis points east, the positive y-axis points north, and the positive z-axis points towards the sky. The figures below illustrate the quadcopter and these maneuvers.

We can represent each of these rotations, that are linear transformations, as matrices that operate on the location vector of the quadcopter, $\mathbf{r}$, to position it at it’s new location. The matrices $R_x(\theta)$, $R_y(\psi)$, and $R_z(\phi)$ represent rotations about the x-axis, y-axis, and z-axis, respectively. The matrices are:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \\ R_y(\psi) = \begin{bmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{bmatrix}, \\ R_z(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$.

(a) Professor Boser wants to make the quadcopter to rotate first by 30° about the x-axis, and then by 60° about the z-axis. Use $R_x(\theta)$, $R_y(\psi)$, and $R_z(\phi)$ to construct a matrix that performs the operations in the specified order. You may use an ipython notebook for algebra, but show in your solutions the matrices and the operations you are doing on them by hand.

(b) Professor Boser accidentally punched in the two commands in reverse. The rotation about the z-axis occurred before the rotation about the x-axis. Use $R_x(\theta)$, $R_y(\psi)$, and $R_z(\phi)$ to construct a matrix that performs the operations that accidentally happened. You may use an ipython notebook for algebra. Write out the matrices you are multiplying, and the computed matrix.

(c) Say the quadcopter was at $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Where did Professor Boser intend for the quadcopter to end up? Where did it actually end up? Are they the same?

(d) Say the quadcopter starts out at distance 1 from the origin, i.e. $||\mathbf{r}|| = 1$. Say Professor Boser performs an arbitrary sequence of operations (unknown to you) using the three transformation matrices $R_x(\theta)$, $R_y(\psi)$, and $R_z(\phi)$. Each transformation could be used more than once. Then the quadcopter ends up at location $\mathbf{s}$. What is $||\mathbf{s}||$?

4. Image Stitching
Learning Objective: This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera’s field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using “image stitching”. Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. It’s your job to figure out how to stitch the images together using Marcela’s common points to reconstruct the larger image.

![Figure 0](image1)

Figure 0: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $R$, and a vector, $\vec{T}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $R$ and $\vec{T}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image and $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

(1)
This problem focuses on finding the unknowns (i.e. the components of $R$ and $\vec{T}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $R$ and vector $\vec{T}$ transform a vector, $\vec{v}_0$, consider this similar equation,

$$\vec{v}_2 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{v}_0 + \vec{v}_1.$$  \hspace{1cm} (2)

Using $\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is $\vec{v}_2$? On a single plot, draw the vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2$ in two dimensions. Describe how $\vec{v}_2$ is transformed from $\vec{v}_0$ (e.g. rotated, scaled, shifted).

(b) Multiply Equation (1) out into two scalar linear equations. What are the known values and what are the unknowns in each equation? How many unknowns are there? How many independent equations do you need to solve for all the unknowns? How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns?

(c) What is the vector of unknown values? Write out a system of linear equations that you can use to solve for the unknown values (you should use multiple pairs of points $\vec{p}$’s and $\vec{q}$’s to have enough equations based on what you found in part b). Transform these linear equations into a matrix equation, so that we can solve for the vector of unknown values.

(d) In the IPython notebook prob3.ipynb, you will have a chance to test out your solution. Plug in the values that you are given for $p_x, p_y, q_x$, and $q_y$ for each pair of points into your system of equations to solve for the matrix, $R$, and vector, $\vec{T}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. You are not responsible for understanding the image stitching code or Marcela’s algorithm.

(e) We will now explore when this algorithm fails. Marcela’s algorithm gives us a new set of corresponding points between the images (Figure 1), but the new points are collinear (i.e. we can draw a straight line between them in the image). Marcela suspects that you will run into an issue when solving for your unknown vector. In the IPython notebook prob3.ipynb, try to plug in the new corresponding points and solve for new unknown values. What do you expect to happen? Explain why?

Figure 1: Two images to be stitched together with collinear pairs of matching points labeled.

(f) **(PRACTICE)** Now more generally, explain why if $\vec{p}_1, \vec{p}_2, \vec{p}_3$ are **collinear**, the system of equations you created in part (c) will have an infinite number of solutions.

**Fact:** Vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$, are collinear when the points $P_1(p_{1x}, p_{1y}), P_2(p_{2x}, p_{2y}), P_3(p_{3x}, p_{3y})$ that correspond to the heads of the vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$, form a straight line. This can be formally expressed by the condition: $(\vec{p}_2 - \vec{p}_1) = k(\vec{p}_3 - \vec{p}_1)$ for some $k \in \mathbb{R}$.
5. Properties of Pump Systems

**Learning Objectives:** This problem illustrates how matrices and vectors can be used to represent linear transformations. It also foreshadows concepts covered next week in class, where we will be exploring matrix inversion. It turns out that matrix inversion is closely related to ideas of linear dependence and independence, which you can use in this problem.

Throughout this problem, we will consider a system of reservoirs connected to each other through pumps. An example system is shown below in Figure 2, represented as a graph. Each node in the graph is marked with a letter and represents a reservoir. Each edge in the graph represents a pump which moves a fraction of the water from one reservoir to the next at every time step. The fraction of water is written on top of the edge.

![Figure 2: Pump system](image)

(a) Consider the system of pumps shown above in Figure 2. Let $x_a[n]$ and $x_b[n]$ represent the amount of water in reservoir $a$ and $b$, respectively, at time step $n$. Find a system of equations that represents every $x_i[n+1]$ in terms of all the different $x_i[n]$.

(b) For the system shown in Figure 2, find the associated state transition matrix. That is find the matrix $A$ such that:

$$\vec{x}[n+1] = A\vec{x}[n], \text{ where } \vec{x}[n] = \begin{bmatrix} x_a[n] \\ x_b[n] \end{bmatrix}$$

(c) Suppose that the reservoirs are initialized to the following water levels: $x_a[0] = 0.5, x_b[0] = 0.5$. In a completely alternate universe, the reservoirs are initialized to the following water levels: $x_a[0] = 0.3, x_b[0] = 0.7$. For both initial states, what are the water levels at timestep 1 ($\vec{x}[1]$)? Use your answer from part (b) to compute your solution.

(d) If you observe the reservoirs at timestep 1, can you figure out what the initial ($\vec{x}[0]$) water levels were? Why or why not?

(e) Now let us generalize what we observed. Say there is a transition matrix $A$ representing a pump system. Say there exist two distinct initial state vectors $\vec{x}[0]$ and $\vec{y}[0]$ (i.e. water levels) that lead to the same state vector $\vec{x}[1]$ after $A$ acts on them. You do not know which of the two initial state vectors you started in. Can you decide which initial state you started in by observing $\vec{x}[1]$? What does this say about the matrix $A$?

(f) Set up the state transition matrix $A$ for the system of pumps shown below. Compute the sum of the entries of the columns of the state transition matrix. Is it greater than/less than/equal to 1? Explain what this $A$ matrix physically implies about the total amount of water in this system.  

*Note:* If there is no “self-arrow/self-loop,” you can interpret it as a self-loop with weight 0, i.e. no water returns.
6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID’s. (In case of homework party, you can also just describe the group.) How did you work on this homework?