EECS 16A Fall 2019 — Designing Information Devices and Systems I — Homework 1

This homework is due Friday September 6, 2019, at 23:59.
Self-grades are due Monday September 9, 2019, at 23:59.

Please note that future homeworks will have more problems than HW1 and we ask you to plan accordingly.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

Please attach a PDF of your Jupyter notebook for all the problems that involve coding. Make sure the results of your plots (if any) are visible. Please assign the PDF of the notebook to the correct problems on Gradescope — we will be unable to grade the problems without this assignment or submission.

Homework Learning Goals: *The objective of this homework is to introduce systems of linear equations. This homework additionally serves as an introduction to working with the Python environment through IPython/Jupyter notebooks.*

1. Counting Solutions

Learning Goal: *(This problem is meant to illustrate the different types of systems of equations. Some have a unique solution and others have no solutions or infinitely many solutions. We will learn in this class how to systematically figure out which of the three above cases holds.)*

For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there are an infinite number of solutions, describe the set of solutions. If there is no solution, explain why. **Show your work**.

Solution: For each of these solutions, two different approaches to the solution are shown. At this point in class, we only expect you to be able to follow the first approach. The second approach will be explained to you in the course of the next weeks — please do not worry at all if the notation and the style of the solution are unfamilar — this only serves as a preview of what is to come.

(a)

$$\begin{aligned} 2x &+ 3y &= 5 \\ x &+ y &= 2 \end{aligned}$$

Solution:

Solution A:

$$2x + 3y = 5 \tag{1}$$

$$x + y = 2 \tag{2}$$

Subtract: (1) - 2*(2)

$$y = 1 \tag{3}$$

Now we plug in (3) into (2) and solve for x

$$\begin{aligned} x + 1 &= 2 \\ \rightarrow \quad x &= 1 \end{aligned} \tag{4}$$

From (3) and (4), we get the unique solution:

$$\begin{aligned} x &= 1 \\ y &= 1 \end{aligned}$$

Solution B:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \longleftarrow R_1 - 2R_2 \mapsto R_1$$
$$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right] \longleftarrow R_2 - R_1 \mapsto R_2$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)

$$\begin{array}{ccccccc} x & + & y & + & z & = 3 \\ 2x & + & 2y & + & 2z & = 5 \end{array}$$

Solution:
Solution A:

$$x + y + z = 3 \tag{5}$$
$$2x + 2y + 2z = 5 \tag{6}$$

Subtract: (6) - 2*(5)

$$0 = -1 \tag{7}$$

We see this results in an inconsistency in (7), indicating that no values of x,y,z can satisfy both equations. Therefore there are no solutions.

Solution B:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \longleftarrow R_2 - 2R_1 \mapsto R_2$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are inconsistent and we have a row of zero equating to a nonzero value. In other words, no values of x, y, and z can satisfy both equations simultaneously.

(c)

$$\begin{aligned} - y + 2z &= 1 \\ 2x + z &= 2 \end{aligned}$$

Solution:

Solution A:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

We notice that because we cannot cancel out x or y using the other equation, the equations do not contradict each other so there must exist an infinite number of solutions. We choose z to be our free variable and can then solve each equation in terms of z.

$$y = 2z - 1$$
$$x = 1 - \frac{1}{2}z$$

Solution B:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all. Using Gaussian Elimination, we can add an additional equation which provides no unique information

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives us the square matrix formulation we were first introduced to, and can see after rearranging the matrix into upper triangular form that we have a zero pivot. That said, the equations do not contradict each other, so we can find the set of solutions:

$$y = 2z - 1$$
$$x = 1 - \frac{1}{2}z$$

Infinite solutions, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t + 1 \\ 2t - 1 \\ t \end{bmatrix} \forall t \in \mathbb{R}$

(d)

$$\begin{aligned} x + 2y &= 3 \\ 2x - y &= 1 \\ 3x + y &= 4 \end{aligned}$$

Solution:

Solution A:

In this case, there are three equations with only two unknowns. However, this fact alone does not tell us whether there is a unique solution, no solution, or an infinite number of solutions.

$$\begin{aligned} x \;+2y \;&= 3 && (8)\\ 2x - y \;&= 1 && (9)\\ 3x + y \;&= 4 && (10)\end{aligned}$$

Adding (8) and (9), we obtain

$$3x + y = 4 \quad (11)$$

Notice that equation (11) = equation (10)! Put another way, equation (10) provides no new information about the system that equations (8) and (9) could not tell us (importantly, it also does not contradict any information from the previous equations as well). Knowing this, we focus only on (8) and (9).

Add (8) + 2*(9)

$$\begin{aligned} 5x &= 5\\ \rightarrow \quad x &= 1 && (12)\end{aligned}$$

Plugging this value of x back into (8), we obtain

$$\begin{aligned} 1 + 2y &= 3\\ \rightarrow \quad y &= 1 && (13)\end{aligned}$$

Yielding the unique solution

$$\begin{aligned} x &= 1\\ y &= 1\end{aligned}$$

Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 3\\ 2 & -1 & 1\\ 3 & 1 & 4\end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 4\\ 2 & -1 & 1\\ 3 & 1 & 4\end{array}\right] \longleftarrow R_1 + R_2 \mapsto R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 3 & 1 & 4\\ 2 & -1 & 1\\ 0 & 0 & 0\end{array}\right] \longleftarrow R_3 - R_1 \mapsto R_3$$

$$\rightarrow \left[\begin{array}{cc|c} 3 & 1 & 4\\ 5 & 0 & 5\\ 0 & 0 & 0\end{array}\right] \longleftarrow R_1 + R_2 \mapsto R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1\\ 1 & 0 & 1\\ 0 & 0 & 0\end{array}\right] \longleftarrow R_1 - \frac{3}{5}R_2 \mapsto R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{array}\right] \longleftarrow \text{switch } R_1, R_2$$

Unique solution, $\begin{bmatrix} x\\ y\end{bmatrix} = \begin{bmatrix} 1\\ 1\end{bmatrix}$. Notice how even with the redundant equation, we still have enough information to uniquely find x and y! If this is unclear, the system of linear equations at the end of the

Gaussian Elimination above simply reads out

$$x = 1$$
$$y = 1$$
$$0 = 0$$

Do not add new columns! Because each column represents a variable, we should not add columns of zeros in an attempt to reshape our matrix into a square. In this problem, the first equation does not provide any new information, but no more information is necessary to solve for x and y.

(e)

$$\begin{array}{ccccl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

Solution:
Solution A:

$$x + 2y = 3 \tag{14}$$
$$2x - y = 1 \tag{15}$$
$$x - 3y = -5 \tag{16}$$

Add: (14) + (16)

$$2x - y = -2 \tag{17}$$

Subtract: (15) - (17)

$$0 = 3$$

This is an inconsistent equation, so there is no solution.

Despite the fact that there were more equations than unknowns, this did not tell us anything about the solutions of the system.
Solution B:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 2 & -1 & -2 \end{array}\right] \longleftarrow R_3 + R_1 \mapsto R_3$$
$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 0 & 0 & 3 \end{array}\right] \longleftarrow R_2 - R_3 \mapsto R_3$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = 3$ would need to be true. Even though we have more equations than unknowns, that does not guarantee that a unique solution, or any solutions, exist.

2. Filtering Out The Troll

Solution: `#SystemsOfEquations` `#LinearCombination`

Learning Goal: *(The goal of this problem is to represent a practical scenario using a simple model of directional microphones. Students will tackle the problem of sound reconstruction through solving a system of linear equations. This also introduces the idea of a linear combination, i.e. a weighted sum of quantities.)*

You attended a very important public speech and recorded it using a recording device that consists of two directional microphones. However, there was this particular person in the audience who was trolling around, adding noise to the recording. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll's noise and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's noise is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.

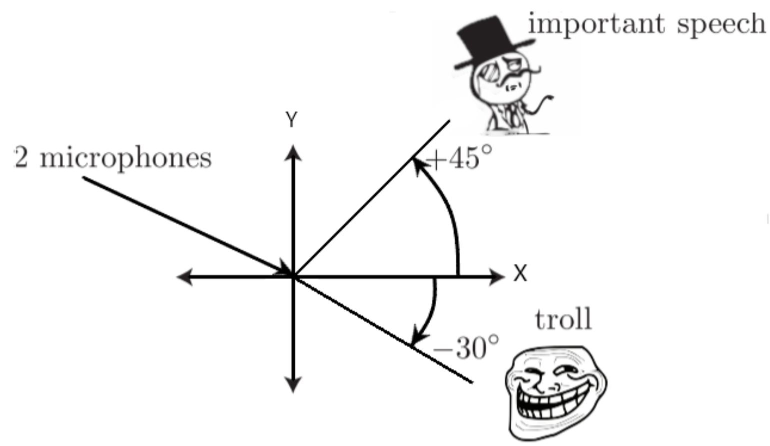

Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on where they are coming from. For the first microphone, when a signal is coming from an angle θ with respect to the x-axis, it is weighted by the factor $f_1(\theta) = \cos(\theta)$. If there are two signals simultaneously playing (as is the case with the speech and the troll noise), then both are recorded as a linear combination, each weighted by the respective $f_1(\theta)$ for their angles). For the second microphone, if the signal is coming from an angle θ with respect to the x-axis, then the signal is weighted by the factor $f_2(\theta) = \sin(\theta)$. The linear combination also applies to the second microphone. Graphically, the directional characteristics of the two microphones are given in Figure 2.

We can now refer to the diagram in Figure 1 and develop a mathematical model of the microphone recordings. Let the person who gave the important speech and the troll be speakers A and B, respectively. The person who gave the important speech (speaker A) was located at angle $\alpha = +45°$ relative to the x-axis, and the troll (speaker B) was located at angle $\beta = -30°$ relative to the x-axis. Speaker A produced an audio signal represented by the vector $\vec{a} \in \mathbb{R}^n$. That is, the i-th entry of vector $\vec{a}$ was the signal at the i-th time step. Similarly, speaker B produced an audio signal $\vec{b} \in \mathbb{R}^n$, where the i-th entry of vector $\vec{b}$ was the signal at the i-th time step.

Therefore, the first microphone recorded the signal

$$\vec{m}_1 = f_1(\alpha) \cdot \vec{a} + f_1(\beta) \cdot \vec{b},$$

and the second microphone recorded the signal

$$\vec{m}_2 = f_2(\alpha) \cdot \vec{a} + f_2(\beta) \cdot \vec{b}.$$

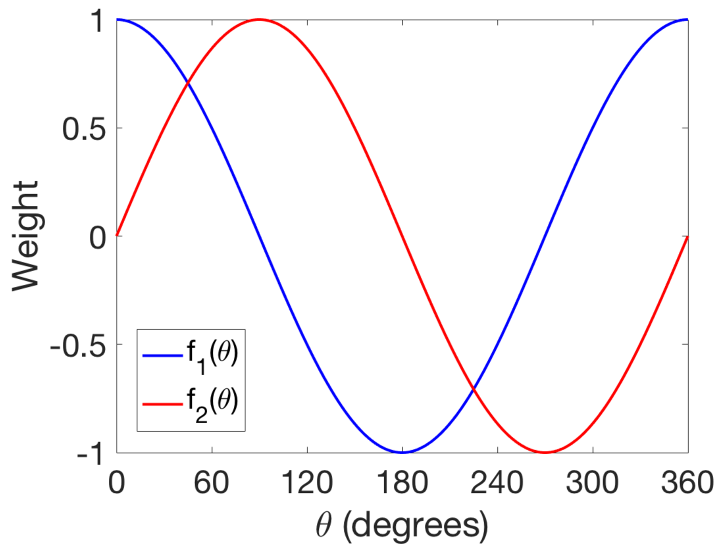

Figure 2: Weights for recorded audio signals for each of the two microphones, as a function of audio source angle θ. Microphone 1 is blue and microphone 2 is red. Note that a weight can be negative as well as positive.

(a) Using the notation above, express the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ (i.e. the signals recorded by the first and the second microphones, respectively) as a linear combination of $\vec{a}$ and $\vec{b}$.

Solution:

$$\begin{aligned}\vec{m}_1 &= \cos\left(\frac{\pi}{4}\right) \cdot \vec{a} + \cos\left(-\frac{\pi}{6}\right) \cdot \vec{b} \\ &= \frac{1}{\sqrt{2}} \cdot \vec{a} + \frac{\sqrt{3}}{2} \cdot \vec{b} \\ \vec{m}_2 &= \sin\left(\frac{\pi}{4}\right) \cdot \vec{a} + \sin\left(-\frac{\pi}{6}\right) \cdot \vec{b} \\ &= \frac{1}{\sqrt{2}} \cdot \vec{a} - \frac{1}{2} \cdot \vec{b}\end{aligned}$$

(b) Recover the important speech $\vec{a}$, as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{a} = u \cdot \vec{m}_1 + v \cdot \vec{m}_2$ (where u and v are scalars). What are the values of u and v?

Solution:

Solving the system of linear equations yields

$$\vec{a} = \frac{\sqrt{2}}{1+\sqrt{3}} \cdot \left(\vec{m}_1 + \sqrt{3}\vec{m}_2\right).$$

Therefore, the values are $u = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $v = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $\vec{b} = \frac{2}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2)$. Substituting $\vec{b}$ back into the second equation and multiplying through by $\sqrt{2}$ gives that $\vec{a} = \sqrt{2}(\vec{m}_2 + \frac{1}{\sqrt{3}+1}(\vec{m}_1 - \vec{m}_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the trolls contribution. Once we have the troll contribution, we can remove it and obtain the important speakers sole content.

(c) Partial IPython code can be found in `prob1.ipynb`. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol1.ipynb`. The speaker says: "All human beings are born free and equal in dignity and rights." and the speech was taken from the Universal Declaration of Human Rights.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

3. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.