• EECS 16A

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Lecture 5

Announcements

- Discussion sections. – new one + extended sectio
- HW solutions in OH.

- One more proof.
- Linear transformations
- Matrix-matrix multiplication.

"Without loss of generality" WLOG

→ 3 numbers: a, b, c.

WLOG, let a be the largest number.

c be the smallest number

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Thm: If $A\vec{x} = \vec{b}$ has two or more solutions, then $\Longrightarrow$ columns of A are linearly dependant.

Proof:

Let $\vec{x_1}$ and $\vec{x_2}$ be two distinct solutions

$A\vec{x_1} = \vec{b}$ AND $A\cdot\vec{x_2} = \vec{b}$

$\Longrightarrow A\vec{x_1} = A\vec{x_2}$

$\Longrightarrow A(\vec{x_1} - \vec{x_2}) = 0$

Let $\vec{y} = \vec{x_1} - \vec{x_2} \neq 0$

$$A\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = 0 \quad , \quad A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \end{bmatrix}$$

$y_1\cdot\vec{a_1} + y_2\cdot\vec{a_2} + \cdots y_n\cdot\vec{a_n} = 0$

Without loss of generality, let $y_1 \neq 0$

$$\vec{a_1} = -\frac{y_2}{y_1}\cdot\vec{a_2} + \cdots + \frac{-y_n}{y_1}\vec{a_n}$$

∎ QED.

To show: For some i

$$\vec{a_i} = \sum_{\substack{j=1 \\ j\neq i}}^{n} c_j\cdot\vec{a_j}$$

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Second definition: Linear dependance.

$\{\vec{a_1}, \vec{a_2}, \dots \vec{a_n}\}$ are linearly dependant if there exist

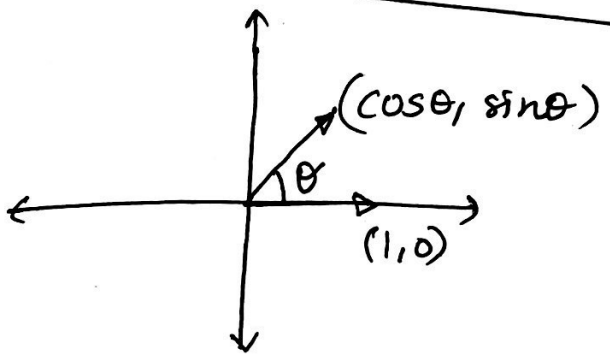
$c_1, c_2 \cdots c_n$, not all zero.

such that $c_1\vec{a_1} + c_2\vec{a_2} + \cdots c_n\vec{a_n} = 0$.

- Linear transformations: Matrices are linear operators.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

A



"Rotation matrix"

What about $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ "Reflection matrix"

$$\begin{bmatrix} 13 & 53 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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• Matrix is a linear transformation.

• f is a linear transformation.

if $f(x+y) = f(x) + f(y)$. "presering addition"

$f(\alpha \cdot x) = \alpha \cdot f(x)$. "scalar multiplication"

↑ Constant

Matrix A.

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

$A(\alpha \cdot \vec{x}) = \alpha \cdot (A\vec{x})$.

e.g. $f(\vec{x}) = 2\vec{x}$

- $f(\vec{x}+\vec{y}) = 2(\vec{x}+\vec{y}) = 2\vec{x} + 2\vec{y} = f(\vec{x}) + f(\vec{y})$
- $f(\alpha \cdot \vec{x}) = 2 \cdot \alpha \cdot \vec{x} = \alpha \cdot f(\vec{x})$.

LINEAR.

eg. $g(x) = x^2$.

$f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$.

$f(x) + f(y) = x^2 + y^2$

NOT LINEAR.

~~$f(x) =$~~

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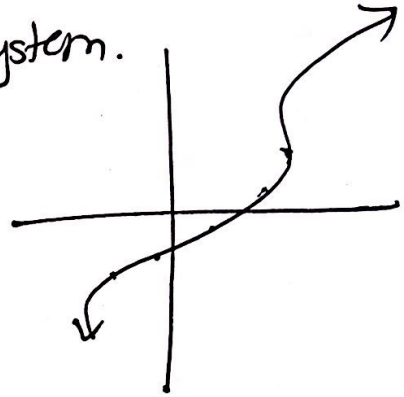
State transformations

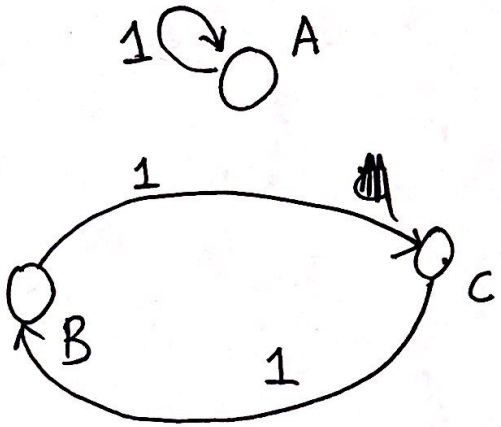
Vectors are often used to represent the state of a system.

e.g. "state" of a car. $x(t)$, $y(t)$, $v(t)$



Represent: $\text{state} = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$ "state vector" @ time t: $\vec{s}(t)$

System of pumps and reservoirs. — Tanks of water.



$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$x_A(t)$: Water in tank A at time t

$x_B(t)$: ——— B ———

$x_C(t)$: ——— C ———.

Run pump.

What is $\vec{x}(t+1)$?

$$x_A(t) = x_A(t+1)$$
$$x_B(t) = x_C(t+1)$$
$$x_C(t) = x_B(t+1)$$

State transition matrix.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} = \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}$$

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~~$x_A(t)$~~ $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$QQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\vec{x}(t+2)$ in terms of $\vec{x}(t)$

$\vec{x}(t+2000)$ ——— $\vec{x}(t)$

$Q \cdot Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

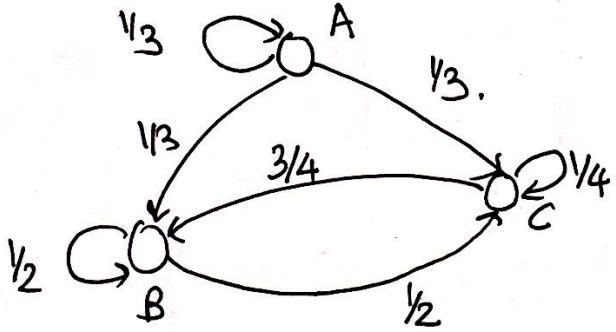
Identity transformation.

$Q \cdot \vec{x}(t) = \vec{x}(t+1)$.

$Q \cdot \vec{x}(t+1) = \vec{x}(t+2)$.

$Q.(Q.\vec{x}(t)) =$ (Q.Q)

Other example:



$x_A(t+1) = \frac{1}{3} x_A(t) + 0 \cdot x_B(t) + 0\, x_C(t)$

$x_B(t+1) = \frac{1}{3} x_A(t) + \frac{1}{2} \cdot x_B(t) + \frac{3}{4} \cdot x_C(t)$

$x_C(t+1) = \frac{1}{3} x_A(t) + \frac{1}{2} \cdot x_B(t) + \frac{81}{4} \cdot x_C(t)$

$\vec{x}(t+1) = Q \cdot \vec{x}(t)$

$= \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1/2 & 3/4 \\ 1/3 & 1/2 & 1/4 \end{bmatrix} \cdot \vec{x}(t)$

inflows to A

outflow from A

$Q.Q.Q \cdots Q$

. . . .

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Matrix-matrix multiplication. — "Stacked" matrix-vector mult.

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = A \cdot \begin{bmatrix} \vec{b_1} & \vec{b_2} \end{bmatrix}$$

$\vec{b_1}$ $\vec{b_2}$

$$= \begin{bmatrix} | & | \\ A\vec{b_1} & A\vec{b_2} \\ | & | \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$A\vec{b_1}$ $A\vec{b_2}$

$$A \cdot B = A \cdot \begin{bmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ A\vec{b_1} & A\vec{b_2} & \cdots & A\vec{b_n} \\ | & | & & | \end{bmatrix}$$

$A \cdot B$
$m \times n$ $n \times r$

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① Matrix - mult. does not commute.

$$A \cdot B \neq B \cdot A.$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$$

② MM is associative

$$(C(B(A\vec{x}))) = (CB)(A\vec{x})$$

$$CBA = (CB) \cdot A = C \cdot (BA)$$

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