Solving under-determined sets of equations

Suppose $Ax = b$

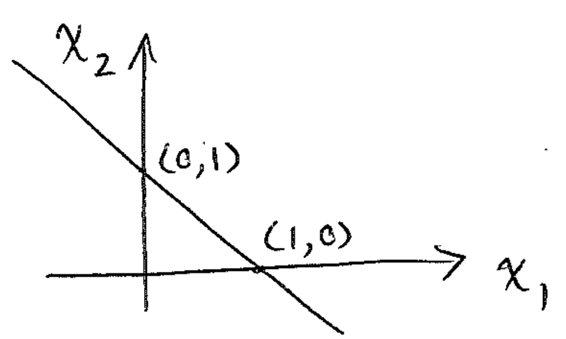
where A is an $n \times m$ matrix
b is a known n-vector
x is an unknown m-vector

Assume $n < m$, that is, there are fewer constraints than unknowns:

ie. $$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

We will also assume A has full row rank ($\text{rank } A = n$).

example 1



line in the plane.
How to represent as $Ax = b$?

In general, in this case,

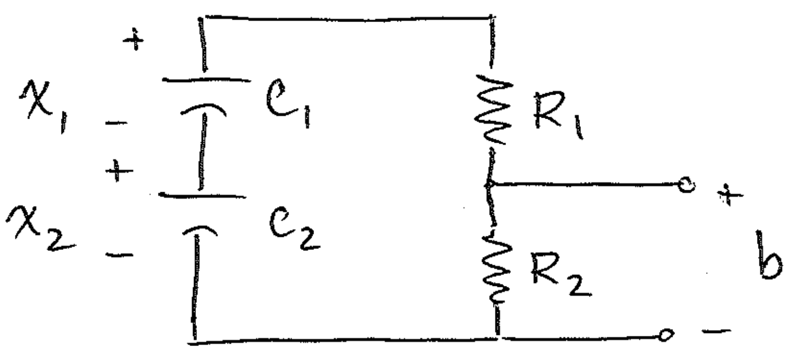
$$Ax = b$$

has an infinite number of solutions.

We can pick one of these solutions by finding the smallest one:

$$\min_x \|x\|^2 \quad \text{s.t. } Ax = b.$$

example 2



Suppose C_1, C_2, R_1, R_2 are given, and b is a desired output voltage.

Find the capacitor voltages x_1 & x_2 to minimize $\|x\|^2$.

We solve this constrained optimization problem using the method of <u>Lagrange multipliers</u>, in which we add a term to the quantity to be minimized:

$$\min_{x,\lambda} \quad \|x\|^2 + \lambda^T(b - Ax) \qquad (*)$$

- Differentiating(*) wrt x and setting the result to zero:

$$\frac{\partial}{\partial x}\left(x^T x + \lambda^T(b - Ax)\right) = 0$$

$$2x^T - \lambda^T A = 0$$

$$\therefore\ 2x - A^T\lambda = 0$$

premultiply by A:

$$2Ax - AA^T\lambda = 0$$

$$\therefore\ \lambda = (AA^T)^{-1}\, 2Ax. \qquad (1)$$

- Differentiating(*) wrt λ and setting the result to zero:

$$Ax = b \qquad (2)$$

$$\therefore \quad \lambda = (AA^T)^{-1} 2b$$

and since $\quad 2x - A^T\lambda = 0$

$$\boxed{x = A^T (AA^T)^{-1} b.}$$

Least-norm sol$^{\underline{n}}$ to $Ax = b$.

<u>example 3</u> Find the least norm solution to

$$x_1 + x_2 + x_3 = 1$$
$$-x_1 - x_2 + x_3 = 0$$