
EE16A: Lecture 3

Introduction to Vectors

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References

The material in this lecture was drawn from:

1. S. Boyd and L. Vandenberghe, “Introduction to Matrix Methods and Applications”, Stanford, October 2014.
Available at: <http://stanford.edu/class/ee103/mma.pdf>
2. L. El Ghaoui, “Optimization Models and Applications”, Livebook, 2015. Available at:
<http://livebooklabs.com/keepies/c5a5868ce26b8125>
3. G. Strang, “Linear Algebra and its Applications”, 4th edition, 2005.

Outline

- Vectors
 - Examples
 - Zero and unit vectors
 - Vector addition
 - Scalar multiplication
 - Inner product, norm, angle

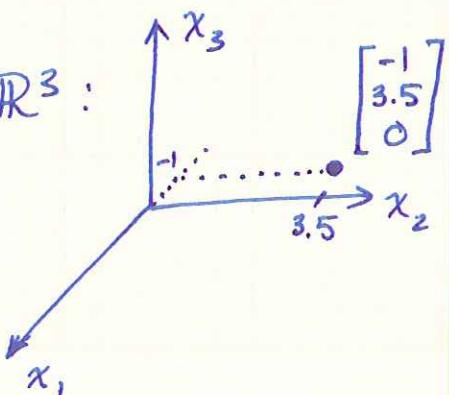
Suppose we are given a collection of n real numbers, x_1, x_2, \dots, x_n ; for example, for $n=3$, we could have $-1, 3.5, 0$.

We can represent this collection as a single point in an n -dimensional space, denoted:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Defⁿ $x \in \mathbb{R}^n$ where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is called a vector. In this case, we say it is a vector in \mathbb{R}^n . Each real number x_i is called a component, or element, of the vector.

Example $\begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} \in \mathbb{R}^3$ is a vector in \mathbb{R}^3 , and represents a single point in \mathbb{R}^3 :



The size of a vector is the number of components it contains.

Two vectors x and y are said to be equal, $x = y$, if they have the same size, and $x_i = y_i$ for all i .

Remarks Vectors can be much more general than the popular $x \in \mathbb{R}^n$. For example,

(a) $x \in \mathbb{C}^n$; here x_i is a complex number

$$x_i = \Delta_i + j\omega_i$$

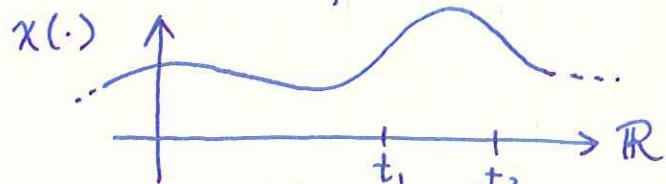
(b) $x \in \mathbb{R}^{n \times n}$; here x is an $n \times n$ matrix with real entries:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}, x_{ij} \in \mathbb{R}.$$

(we'll see more of this soon!)

(c) functions are also vectors!

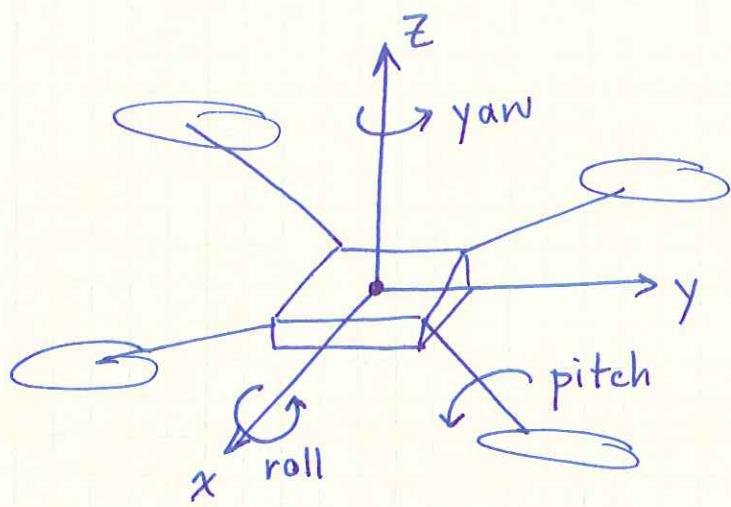
- for example: x could be a continuous function over the real line



- We usually write $x(t)$ as "x as a function of t", where $t \in \mathbb{R}$ and $x(t_1), x(t_2)$ etc. as particular values

Example (quadrotor model)

The 3D position, angle, velocity, and angular velocity of a quadrotor at a particular time can be represented as a vector in \mathbb{R}^{12} :



$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\text{roll}} \\ \dot{\text{pitch}} \\ \dot{\text{yaw}} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\text{roll}} \\ \ddot{\text{pitch}} \\ \ddot{\text{yaw}} \end{bmatrix}$$

where $\dot{\cdot}$ represents the derivative with respect to time.

Example (color)

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can represent a color; with its components giving the (R)ed, (G)reen, and (B)lue (RGB) intensity values, $\mathbf{x} \in \mathbb{R}^3$

Example $x \in \mathbb{R}^N$ could represent the sample values of a quantity at N time points; for example, for a car moving along a line, the positions at t_1, t_2, \dots, t_N

$$X = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_N) \end{bmatrix}.$$

Here, $X \in \mathbb{R}^N$ is the vector and $x(t_i)$ represents the position at time t_i .

Example (image).

A black and white image of $M \times N$ pixels can be represented by a vector of length MN , with the components giving grey scale levels at the pixel location, ordered column-wise or row-wise:

$$M \begin{array}{|c|c|c|c|} \hline & \square & \square & \square \\ \hline \end{array} \Rightarrow \begin{bmatrix} \square \\ \square \\ \vdots \\ \square \end{bmatrix} \in \mathbb{R}^{MN}$$

Question: What about a color image?

Question: What about a color video?

Defⁿ

A zero vector is a vector with all the components equal to zero, usually just represented as 0 , where its size is implied from context:

if $x \in \mathbb{R}^n$, then $x + 0 = x$

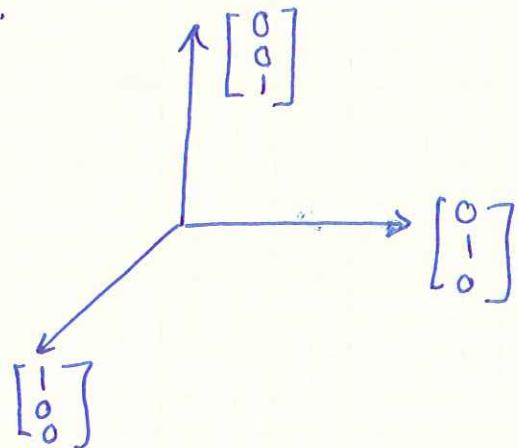
↑ vector addition, we'll talk about this soon

here, it is understood that $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.

Defⁿ A unit vector is one with all components equal to zero, except one element which is equal to 1.

We denote, for example, the three unit vectors in \mathbb{R}^3 as:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Vector addition:

Two vectors of the same size can be added together by adding the corresponding components:

$$\begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.5 \\ 3 \end{bmatrix}$$

Properties of vector addition: for $x, y, z \in \mathbb{R}^n$

commutative $x+y = y+x$

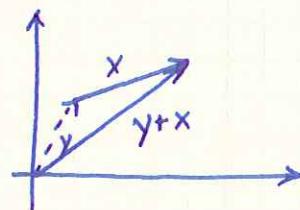
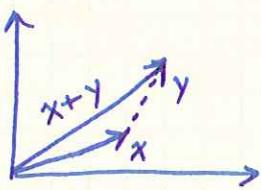
associative $(x+y)+z = x+(y+z)$

zero $x+0 = x$

additive inverse $x+(-x) = 0$

where $-x = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$.

Example



Multiplying a vector by a scalar:

We can also multiply a vector by a number, called a scalar. We'll consider two kinds of scalars:

$$(1) \alpha \in \mathbb{R} \text{ (for } x \in \mathbb{R}^n\text{)}$$

$$(2) \alpha \in \mathbb{C} \text{ (for } x \in \mathbb{C}^n\text{)}$$

We often call this operation scalar multiplication and it is performed by multiplying each component of the vector by the scalar:

$$(-3) \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -10.5 \\ 0 \end{bmatrix}$$

or, in general,

$$\alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Example $-x$, where $x \in \mathbb{R}^n$, is $(-1)x$

↓ vector
 ↑ scalar

Example $0x = 0$

↓ vector
 ↑ scalar ↓ zero vector

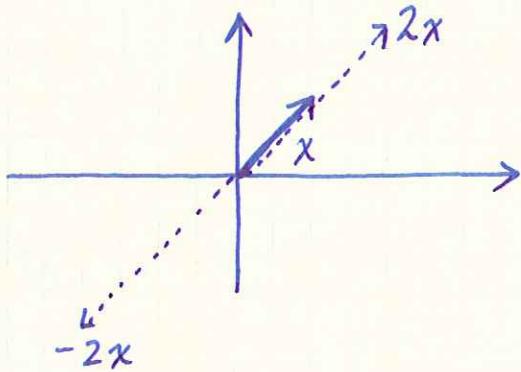
Properties of scalar multiplication: α, β scalars
 $x \in \mathbb{R}^n$

associative: $(\alpha\beta)x = \alpha(\beta x)$

distributive: $(\alpha + \beta)x = \alpha x + \beta x$
 \uparrow
scalar addition \uparrow vector addition

identity: $1 \cdot x = x$

Example:



Defⁿ Inner product (or dot product) between two vectors $x, y \in \mathbb{R}^n$ is:

$$\begin{aligned} x^T y &= [x_1, x_2, \dots, x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n \in \mathbb{R} \\ &= \sum_{i=1}^n x_i y_i \end{aligned}$$

(we also denote this product as $\underset{\text{dot}}{\overset{x}{\cdot}} y$
or $\langle x, y \rangle$)

Note that above we used $()^T$, dot
as in the Transpose of the vector x .

Example: $\begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = -1 + 0 + 0 = -1$

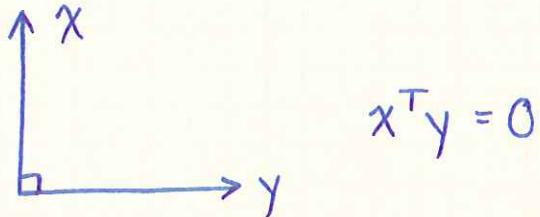
Examples:

Orthogonal vectors

Two vectors x, y are orthogonal if

$$x^T y = 0$$

example (in \mathbb{R}^2):



$$x^T y = 0$$

example (in \mathbb{R}^3):

$$x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$$

$$x^T y = 0 \dots$$

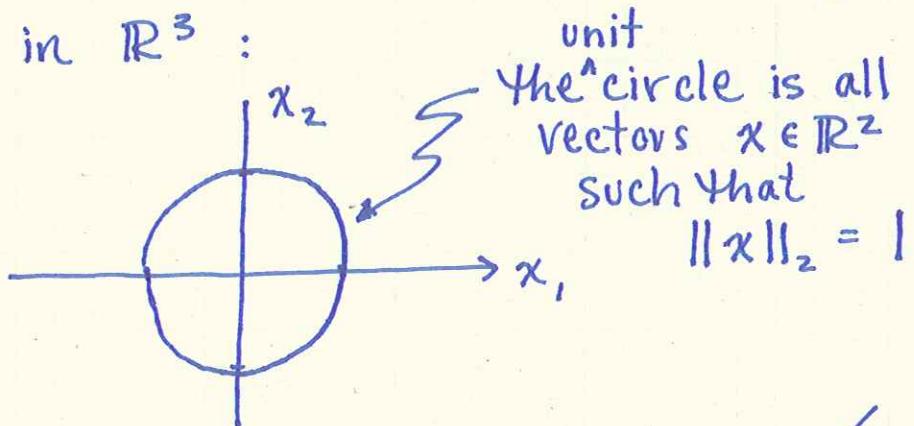
Norms

Defⁿ The Euclidean norm of a vector is given by:

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{x^T x}$$

This corresponds to the usual notion of distance in \mathbb{R}^2 or \mathbb{R}^3

example The set of points with equal Euclidean norm is a circle in \mathbb{R}^2 , or a sphere in \mathbb{R}^3 :



Remark The 2 in the subscript $\|\cdot\|_2$ differentiates the Euclidean norm (or 2-norm, as it is often called) from other useful norms, such as:

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \text{absolute value of } x_i$$

$$\text{or } \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Properties of norms:

$$\|\alpha x\| = |\alpha| \|x\|$$

$$\|x+y\| \leq \|x\| + \|y\| \quad \Delta\text{-inequality}$$

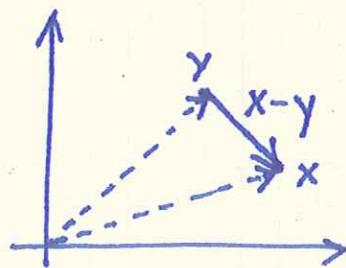
$$\|x\| \geq 0$$

$$\|x\| = 0 \text{ only if } x = 0$$

↑ zero vector

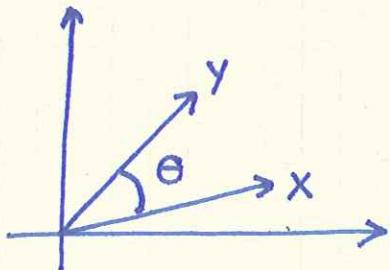
Euclidean distance between two vectors:

$$x, y \in \mathbb{R}^n$$



$\|x-y\|$ is the distance between vectors $x \neq y$

Angle between vectors: $x, y \in \mathbb{R}^n$, both non-zero



$$\theta = \arccos \left(\frac{x^T y}{\|x\| \|y\|} \right)$$

↑ inverse cos, normalized to lie in $[0, \pi]$