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# **EE16A: Lecture 4**

## **Introduction to Matrices and Vector Spaces**

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# References

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The material in this lecture was drawn from:

1. S. Boyd and L. Vandenberghe, “Introduction to Matrix Methods and Applications”, Stanford, October 2014.  
Available at: <http://stanford.edu/class/ee103/mma.pdf>
2. L. El Ghaoui, “Optimization Models and Applications”, Livebook, 2015. Available at:  
<http://livebooklabs.com/keepies/c5a5868ce26b8125>
3. G. Strang, “Linear Algebra and its Applications”, 4<sup>th</sup> edition, 2005.

# *Outline*

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- Matrices:
  - Examples
  - Zero and Identity matrices
  - Transpose and addition
  - Scalar multiplication
  - Matrix-vector multiplication
- Vector Spaces:
  - Examples
  - Bases
  - Coordinates
  - Dimension

Def<sup>n</sup> A matrix is a rectangular array of numbers, written as:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

Each  $A_{ij}$  is a component, or element of the matrix  $A$ , where  $i$  indicates the row index, and  $j$  indicates the column index.

In the example above,  $A$  has 3 rows and 4 columns (and is called a  $3 \times 4$  matrix). In general, we refer to a matrix of size  $m \times n$  as an  $m \times n$  matrix.

If the components of  $A$  are real numbers, we say  $A \in \mathbb{R}^{m \times n}$ .

Two matrices  $A$  and  $B$  are said to be equal,  $A = B$ , if they are of the same size and  $A_{ij} = B_{ij}$  for all  $i, j$ .

- Square matrices
- Tall and Wide matrices.
- Column & row vectors

Example Consider the  $3 \times 2$  matrix A:

$$A = \begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix}$$

This matrix has two column vectors:

$$a_1 = \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 2 \\ -0.1 \end{bmatrix}$$

and three row vectors:

$$b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad b_2 = \begin{bmatrix} 3.5 \\ 2 \end{bmatrix} \quad b_3 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}$$

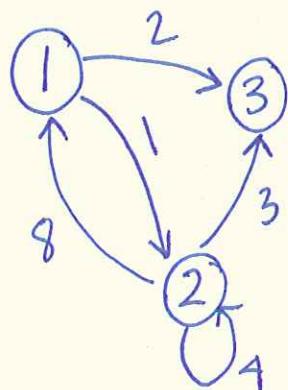
A can be written as:

$$A = [a_1 \ a_2] \quad \text{or} \quad A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix}$$

### Examples:

(a) Images: a black and white image with  $m \times n$  pixels can be represented as an  $m \times n$  matrix, where each component is the greyscale value corresponding to that pixel in the image.

(b) Matrix representation of a network:



$$\begin{bmatrix} 0 & 1 & 2 \\ 8 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Def<sup>n</sup> A zero matrix is a matrix with all the components equal to zero, usually just represented as 0, where its size is implied from context.

Def<sup>n</sup> An identity matrix is a square matrix whose diagonal elements are 1 and whose off-diagonal elements are all 0:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

(note that the column vectors of an identity matrix are the unit vectors)

## Matrix transpose:

The transpose of an  $m \times n$  matrix  $A$ , denoted  $A^T$ , is the  $n \times m$  matrix given by  $(A^T)_{ij} = \underbrace{A_{ji}}_{\substack{\text{ij}^{\text{th}} \text{ component} \\ \text{of } A^T}} = \underbrace{A_{ji}}_{\substack{\text{ji}^{\text{th}} \text{ component of} \\ A}}$ .

## Symmetric square matrices:

A square matrix is symmetric if

$$A = A^T$$

(which means that  $A_{ij} = A_{ji}$  for all  $i, j$ )

## Matrix addition:

Two matrices of the same size can be added together by adding the corresponding components:

$$\begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & -2 \\ 3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2.5 & 0 \\ 3 & 0 \end{bmatrix}$$

## Properties of matrix addition:

for  $A, B, C \in \mathbb{R}^{m \times n}$

commutative  $A + B = B + A$

associative  $(A + B) + C = A + (B + C)$

zero  $A + 0 = A$

additive inverse  $A + (-A) = 0$

## Multiplying a matrix by a scalar:

As with vectors, multiply each component of the matrix by the scalar:

$$(-3) \begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix} = \begin{bmatrix} 3 & -9 \\ -10.5 & -6 \\ 0 & 0.3 \end{bmatrix}$$

example

$-A$  is  $(-1)A$

example

$$\underset{\substack{\uparrow \\ \text{scalar}}}{0} \underset{\substack{\leftarrow \\ \text{matrix}}}{A} = \underset{\substack{\uparrow \\ \text{zero matrix}}}{0}$$

## Properties of scalar multiplication:

$\alpha, \beta$  scalars ( $\mathbb{R}$ )

$A \in \mathbb{R}^{m \times n}$

associative

$$(\alpha\beta)A = \alpha(\beta A)$$

distributive

$$(\alpha + \beta)A = \underset{\substack{\uparrow \\ \text{scalar addition}}}{\alpha A} + \underset{\substack{\uparrow \\ \text{vector addition}}}{\beta A}$$

identity

$$1 \square A = A$$

## Matrix - vector multiplication:

$$A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n$$

$$Ax = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n \\ \vdots \\ \vdots \\ A_{m1}x_1 + \dots + A_{mn}x_n \end{bmatrix}$$

for example:

$$\begin{bmatrix} -1 & 3 \\ 3.5 & 2 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -0.1 \end{bmatrix}$$

- zero matrix
- identity matrix
- picking out columns : rows using unit vectors

example (Input-output interpretation)

$$y = Ax \quad x \in \mathbb{R}^n \quad y \in \mathbb{R}^m \quad A \in \mathbb{R}^{m \times n}$$

We can think of  $x$  as an input,  $y$  as the output, and  $A$  as the effects of the system

As we've seen, vectors and matrices may, in general, look different from each other, but they follow the same rules for addition, scalar multiplication..

Because we will be using vectors and matrices and their properties a lot, it is useful for us to think about a more general way of classifying them:

To do this, we will define a

Vector Space

## Vector Space

A vector space  $(V, \mathbb{F})$  is a set of vectors  $V$ , and a set of scalars  $\mathbb{F}$  (usually  $\mathbb{R}$  or  $\mathbb{C}$ ), and two operations:

- (i) vector addition (+)
- (ii) scalar multiplication (-)

such that:

(i) Vector addition is:

- associative
- commutative
- There is an identity, the zero vector
- There is an additive inverse

(ii) Scalar multiplication is:

- associative
- commutative
- $1 X = X, X \in V$
- $0 X = 0_{\text{zero vector}}$

and scalar multiplication distributes over vector addition:

$$(\alpha + \beta) X = \alpha X + \beta X. \quad \alpha, \beta \in \mathbb{F}, X, Y \in V$$

$$\alpha(X + Y) = \alpha X + \alpha Y.$$

## Examples (of vector spaces):

(i)  $(\mathbb{F}^n, \mathbb{F})$ , ie  $(\mathbb{R}^n, \mathbb{R})$  or  $(\mathbb{C}^n, \mathbb{C})$

(ii)  $(\mathbb{F}^{m \times n}, \mathbb{F})$ , ie  $(\mathbb{R}^{m \times n}, \mathbb{R})$  or  $(\mathbb{C}^{m \times n}, \mathbb{C})$

: and now we can talk about definitions, features, and properties that are general to vector spaces:

## Linear Independence and Dependence

Given a vector space  $(V, \mathbb{F})$ ,  
 The set of vectors  $\{v_1, v_2, \dots, v_p\}$   
 (where each  $v_i \in V$ ) is said to  
 be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p = 0$$

implies that all the  $\alpha_i$  ( $\alpha_1, \alpha_2, \dots, \alpha_p$ )  
 are all zero.

Conversely, the set is linearly dependent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p = 0$$

and not all of the  $\alpha_i$ 's are zero.

(In other words, one of the vectors can be written as a linear combination of the others.)

### Examples

- Any set of vectors containing the zero vector is linearly dependent

- The vectors

$$x_1 = \begin{bmatrix} 0.2 \\ -7 \\ 8.6 \end{bmatrix} \quad x_2 = \begin{bmatrix} -0.1 \\ 2 \\ -1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ -1 \\ 2.2 \end{bmatrix}$$

are linearly dependent, since

$$x_1 + 2x_2 - 3x_3 = 0$$

- The standard unit vectors in  $\mathbb{R}^n$  are linearly independent.

## Basis

Given a vector space  $(V, \mathbb{F})$ ,  
 a set of vectors  $\{b_1, b_2, \dots, b_n\}$   
 is called a basis of  $V$  if:

(i)  $\{b_1, b_2, \dots, b_n\}$  is linearly independent

(ii) any vector in  $V$  can be written as a linear combination of  $\{b_1, b_2, \dots, b_n\}$ , that is,

That  $v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$   
 for some  $\alpha_i \in \mathbb{F}$ .

## examples :

(1)  $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  is a basis for  $\mathbb{R}^n$

(2)  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$   
 is a basis for  $\mathbb{R}^{2 \times 2}$

(3)  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$   
 is also a basis for  $\mathbb{R}^{2 \times 2}$

## Co-ordinates

Given a vector space  $(V, \mathbb{F})$ , any vector  $x \in V$  may be written as a linear combination of the basis vectors :

$$x = \xi_1 b_1 + \xi_2 b_2 + \dots + \xi_n b_n$$

The scalars  $\xi_1, \xi_2, \dots, \xi_n$  are called the coordinates of  $x$  with respect to the basis  $\{b_1, b_2, \dots, b_n\}$

Fact : Once you choose a basis and pick a vector  $x$  to write in terms of that basis, the coordinates are uniquely defined.

## Remarks :

- (i) A basis of a vector space is not unique
- (ii) If  $\{b_1, b_2, \dots, b_n\}$  is a basis for  $(V, \mathbb{F})$ , Then any other basis also has  $n$  elements. The number of elements in the basis is called the dimension of the vector space.

example: What is the dimension of  $\mathbb{R}^n$ ?

of  $\mathbb{R}^{n \times n}$ ?

of  $\mathbb{R}^{m \times n}$ ?

Can you think of an infinite dimensional vector space?