Imaging

Given a stack of bottles: Milk (M), Juice (J) or Empty (O)

\[
\begin{bmatrix}
M & J & O \\
M & J & O \\
M & O & J
\end{bmatrix}
\]

We can’t "see" directly into the stack but we can shine light at different angles to get a sense of what’s in each row or column.

Suppose milk absorbs 3 units of light
Juice absorbs 2 units of light
Empty bottle absorbs 2 units of light

Let’s assign the amount of light absorbed at each item as \( x \)

\[
\begin{bmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{bmatrix}
\]

If we shine light in a straight line, we can determine the amount of light absorbed as the sum of the light absorbed by each bottle.

The amount of light absorbed by the first row is given by:

\[
\sum_{i=1}^{3} x_{i1} = x_{11} + x_{12} + x_{13} = 5
\]

Likewise, the amount of light absorbed by the 3 columns are given the following:

\[
\sum x_{i1} = 6 \\
\sum x_{i2} = 3 \\
\sum x_{i3} = 6
\]

The diagonals are:

\[
\sum x_{ii} = 6
\]
\[ \sum x_{(4-i)i} = 3 \]

One such configuration that satisfies those equations is:

\[
\begin{bmatrix}
3 & 1 & 1 \\
2 & 1 & 3 \\
1 & 1 & 2
\end{bmatrix}
\]

From here we can deduce that the original stack looked like:

\[
\begin{bmatrix}
M & O & O \\
J & O & M \\
O & O & J
\end{bmatrix}
\]

Reading an image

Suppose we have an image represented as a matrix:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
\]

One approach to read the image would be to multiply it by a matrix of 1’s across the diagonal

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
\]

Suppose we take two measurements at the same time, i.e. \( x_1 + x_2, x_2 + x_3, \) etc.

To do this, we would use a \((n-1) \times n\) matrix of pairs of 1’s across the diagonal. This will result in a \((n-1)\) vector.

\[
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
x_1 + x_2 \\
x_2 + x_3 \\
x_3 + x_4 \\
\vdots \\
x_{n-1} + x_n
\end{bmatrix}
\]

However this will not give us enough information to solve the system since we have less equations than unknowns. So, we will take the measurement of the last pixel as well:

\[
\begin{bmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
x_1 + x_2 \\
x_2 + x_3 \\
x_3 + x_4 \\
\vdots \\
x_{n-1} + x_n
\end{bmatrix}
\]

EECS 16A, Spring 2015, Note 5
Why might we want to use this? Because the photo detector has a threshold, and there may not be enough light to activate the detector with only one pixel.

**Linear Equations**

Suppose we have:

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}
\]

This represents:

\[a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1\]
\[a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2\]
\[a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3\]

When can we solve this system? Only when the coefficient matrix is invertible.

**Image processing**

To blur an image:

\[
\begin{bmatrix}
    1/2 & 1/2 & 0 & 0 \\
    0 & 1/2 & 1/2 & 0 \\
    0 & 0 & 1/2 & 1/2 \\
    0 & 0 & 0 & 1/2
\end{bmatrix}
\begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    0.5 \\
    0.5 \\
    0.5 \\
    0.5
\end{bmatrix}
\]

This will produce a gray image. If we want to produce finer results, we can change our interpolation weights.

\[
\begin{bmatrix}
    0.7 & 0.3 & 0 & 0 \\
    0 & 0.7 & 0.3 & 0 \\
    0 & 0 & 0.7 & 0.3 \\
    0 & 0 & 0 & 0.7
\end{bmatrix}
\begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    0.3 \\
    0.7 \\
    0.3 \\
    0.7
\end{bmatrix}
\]

This will produce an image that interpolates from black to white.

**Vector spaces**

A vector space \((V, F)\) is a set of vectors \(V\), and a set of scalars \(F\) and two operators:

a. Vector addition, +

b. Scalar multiplication, ·
Vector addition:

a. Associative $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

b. Commutative $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

c. Additive identity $\vec{v} + \vec{0} = \vec{v}$

d. Additive inverse $\vec{v}, \vec{w}$ such that $\vec{u} + \vec{w} = \vec{0}$

Scalar multiplication:

a. Associative

b. Commutative

c. Multiplicative identity $1 \cdot \vec{v} = \vec{v}$

d. Zero $0 \cdot \vec{v} = \vec{0}$

And the distributive laws:

a. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

b. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$