Lecture notes by Ena Hariyoshi (02/17/2015)

Examples of Eigenvectors

Every 2×2 linear transformation always has two complex eigenvectors. However, the 2×2 linear transformation can have none, one, or two real eigenvectors.

From end of Note 8 This example has two distinct eigenvectors.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{the eigenvectors are } \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R} \text{ and } \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \beta \in \mathbb{R}$$

The eigenvectors are linearly independent, and form a basis for $\mathbb{R}^2$.

Rotation Consider the transformation that rotates any vector by θ, about the origin. Then, there are eigenvectors only for particular values of $\theta : 0, \pi$, and the corresponding eigenvalues are $1, -1$. It is clear that, for these transformations, any vector is an eigenvector. In this case, we can choose any two basis vectors and call them the eigenvectors of the transformation, with the same eigenvalue. The space defined by these two vectors ($\mathbb{R}^2$) is called the eigenspace.

When θ is any other value, the eigenvectors are complex.

Repeated Eigenvector When there is only one eigenvalue ($\det(A - \lambda I) = 0$ has a double root), the eigenvalue is called a repeated eigenvalue. There may be one eigenvector, or two linearly independent eigenvectors forming a basis for an eigenspace. Here is an example where there is only one eigenvector.

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

(This matrix is in the Jordan canonical form of the linear transformation.)

Reflection Consider the transformation that reflects any vector in $\mathbb{R}^2$ through the line defined by $[2\ 1]^T$. How do we solve for its matrix, A?

We can calculate A by first finding the eigenvalues and the eigenvectors.

First eigenvector: any vector on the line $[2\ 1]^T$ will stay where it is

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_1 = 1$$

Second eigenvector: any vector perpendicular to the line $[2\ 1]^T$, going through the origin, will reflect on the other side of the line

$$v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ \lambda_2 = -1$$

Solve:

$$Av_1 = \lambda_1 v_1 \qquad\qquad Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We will get:

$$A = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$$

Stretching Consider that transformation that stretches the x_1 coordinate by 2, and squashes the x_2 coordinate by 2. What are the eigenvalues and the eigenvectors?

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \lambda_1 = 2 \qquad\qquad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \lambda_2 = \frac{1}{2}$$

Solving as in the previous example, we get:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

This is a diagonal matrix. It is because this linear transformation's eigenvectors are the standard basis of $\mathbb{R}^2$. Linear transformations can be expressed as diagonal matrices in the basis of their eigenvectors.

Rotation of an Image Suppose that a pixel of an image is encoded as a vector in $\mathbb{R}^3$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where (x_1, x_2) encodes the pixel's location in $\mathbb{R}^2$, and x_3 encodes the intensity of the pixel.

Consider the linear transformation that rotates pixels by θ in $\mathbb{R}^2$, and keeps the same value of x_3.

The matrix for rotation by θ in $\mathbb{R}^2$ is this:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

And the matrix for the whole linear transformation is this:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The zeroes and the one in the third row and the third column ensure that x_3 is kept the same through the transformation.

Overview of Touchscreens

The goal of a touchscreen is to detect the position of one of more fingers, and to pass the infromation on to a computer of some sort.

Consider a simplified model in which you only want to extract the coordinate where the one finger approximately is, across one spatial dimension of x. You have sensors that detect e.g. how much pressure $I(x)$ is being applied on each spot. Plotted, a finger will look roughly like a Gaussian. The problem is that the world is noisy, with sun spots and alpha particles, which can disturb the measurements. How do we clear out the noise?

One approach is to do a sum or an integral, such as a moving average. A con of this approach is that it smears the signal. However, touchscreens don't need to be super accurate. In fact, we don't need to do a moving average at all.

What is the simplest thing we can do?

We can use a combination of threshold and maximum. We can detect a touch once $I(x)$ goes above a certain threshold, and say that the finger is located at the x with maximum $I(x)$.

Fun fact: some styluses generate their own signals that the touchscreen detects.

What if we wanted to detect the edges of the finger? For this, we can do a difference or a derivative. The coordinates with the greatest difference are where the slope of the intensity is steepest, and where the edge is most likely to be. Note the definition of a derivative:

$$\frac{\partial I}{\partial x} = \lim_{\Delta x \to \infty} \frac{I(x+\Delta x) - I(x)}{\Delta x}$$

It is a difference, taken to the limit as the intevrals between measurements get smaller.

Lastly, algorithms such as least-squares are used to fit ellipses where the computer thinks the finger is - we will learn least squares later in the class.

Main Types of Touchscreens

Optical Touchscreen The earliest touchscreens were optical touchscreens. You'll find them in museums and places that haven't been upgraded in decades. It consists of a regular screen, with light-emitting diodes lined around the rim. Light detectors lined around the rim detect when a finger blocks a beam of light.

Cons:

- low resolution
- ambient light degrades signal
- can support multitouch but has problems with "shadowing"

Acoustic Touchscreen The acoustic touchscreen was very important for a long time. The surface of glass has acoustic properties; when a finger touches it, the properties change. Sound waves are emitted and detected similarly to the optical touchscreen, in order to detect the changing properties.

Resistive Touchscreen The resistive touchscreen, used in early generation phones, laptops, and monitors, is largely obsolete in those devices but is still used in many other applications. Finger pressure makes an electrical contact in a circuit behind the screen. It differs from a regular old button in that there is an array of contact points.

Cons:

- ~~wears out easily~~
- the materials are durable so that they do not wear out easily, therefore you have to press hard
- the "shadowing" problem with multitouch can be solved but is tricky

Pro:

- really cheap

We will look at the resistive touchscreen in this class to learn concepts such as resistance and charge.

Capacitive Touchscreen The hottest and the greatest tech is the capacitive touchscreen, which has taken over all of the devices you probably use today. When you first used a multimeter, did you try to measure the resistance of your finger? Your skin is mostly insulating. However, the inside of your body is not: its resistance is less than a kilohm. A capacitor is a bucket that holds charge, and your body is a bucket that holds charge. Your finger is a pretty good capacitor! When you touch the screen, your finger bleeds charge, altering the capacitance of the touch position.