Taking a simple case, \( N = 2 \) with no phase shift, we have \( q(t) = \cos(\omega_1 t) x_1(t) + \cos(\omega_2 t) x_2(t) = \frac{x_1}{2} + 2x_1 \cos(2\omega_1 t) + \frac{x_2}{2} \cos((\omega_1 + \omega_2) t) + \frac{x_2}{2} \cos((\omega_1 - \omega_2) t) \). Using \( \omega_1 \) and \( \omega_2 \), we can find a fundamental frequency \( \omega_0 \) such that \( \omega_1 = k_1 \omega_0 \) and \( \omega_2 = k_2 \omega_0 \), for \( k_1, k_2 \in \{0, 1, 2, 3...\} \).

Using the same technique we have several times before. In order to get the messages back out, we take averages to filter out the "noise" that we have. By definition of a continuous average, we have \( \bar{q} = \frac{1}{T_0} \int_0^{T_0} q(t) dt \). This, however doesn’t cancel out the extraneous \( \cos \) terms (note, we only want \( x_1 \) for this example). Conveniently, \( \int_0^b \cos(t) dt = 0 \) when \( a - b = 2k\pi \); that is, the \( \cos \) terms will evaporate if we choose \( a, b \) such that \( b - a \) covers an integer number of periods. Since \( T = \frac{2\pi}{\omega_0} \), we can find a period length \( T_0 = \frac{2\pi}{\omega_0} \), that we can then integrate over. Doing so yields \( \bar{q} = \frac{x_1}{2} + \frac{x_1}{2} T_0 \int_0^{\frac{T_0}{2}} \cos(2\omega_0 t) \ldots dt = \frac{x_1}{2} \). Since all terms inside the integral contain some \( \cos \) term.

### Complex Variables on the Unit Circle

Here, we introduce the concept of a phasor \( q(t) = e^{it} \). Signals can be represented as the sum of complex exponentials \( x(t) = \Sigma x_k e^{i\omega_k t} \).

\( q(t) \) is always on the unit circle

Let \( q(t) \) be a phasor \( q(t) = e^{it} \). It is immediately obvious that \( q(0) = 1 \). Taking the derivative \( \frac{dq}{dt} \), we have \( \frac{dq}{dt} = ie^{it} = iq(t) \). If we let \( a(t) = Re(q(t)), b(t) = Im(q(t)) \), we have \( \frac{da}{dt} = \frac{da(t)}{dt} = i \frac{db(t)}{dt} \). However, we also found earlier that \( \frac{dq}{dt} = iq(t) = -b(t) + ia(t) \), so we have \( \frac{da(t)}{dt} = \frac{db(t)}{dt} = a(t) \). Combining these two together, we arrive at \( \frac{da}{dt} a = -\frac{db}{dt} b \), or \( \frac{da}{dt} a + \frac{db}{dt} b = \frac{1}{2} \frac{d}{dt} [a^2 + b^2] = 0 \). This means that \( a^2 + b^2 = C \) for some constant \( C \), and therefore \( q(t) \) lies on a circle. Since \( q(0) = 1 \), we know that \( a^2 + b^2 = 1 \), so \( q(t) \) lies on the unit circle.

QED