Lecture notes by Nikhil Sharma (04/14/2015)

Euler's Formula / Inverse Formulas

As we have learned before, Euler's formula is given by

$$e^{it} = cos(t) + isin(t)$$

and so for any arbitrary function of t, we can map the motion of a particle on the unit circle. For example, let

$$\theta(t) = 2\pi f_0 t + \theta_0$$

Given this, we can determine that this maps the motion of a particle on the unit circle with initial position at an angle θ_0 to the x-axis traveling counterclockwise at f_0 revolutions per second.

Using Euler's formula, we can derive two inverse formulas, solving for $cos(t)$ and $sin(t)$ respectively that have many applications in signal processing, as we shall soon see. To fulfill this derivation, we compute the following:

$$e^{it} = cos(t) + isin(t)$$
$$e^{-it} = cos(t) - isin(t)$$

Adding (2) and (1) and solving for $cos(t)$ yields

$$cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$$

Similarly, if we subtract (2) from (1), hence eliminating $cos(t)$, and solve for $sin(t)$, we yield

$$sin(t) = \frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}$$

Spectrum

The spectrum of a signal x is a decomposition of x into a linear combination of complex exponentials (e.g. $e^{i\omega t} = e^{i2\pi ft}$). Hence, the application of Euler's inverse formulas comes into play, as signals tend to be send as sine or cosine waves, and can be interpreted as a linear combination of complex exponentials using Euler's inverse formulas.

Decaying a signal into its complex exponential form can be very useful in plotting a frequency diagram, $X(f)$, for an arbitrary function $x(t)$. For example, let

$$x(t) = e^{i2\pi f_0 t}$$

We can see that the frequency of this signal is simply f_0, and so on a frequency diagram, would look as follows:
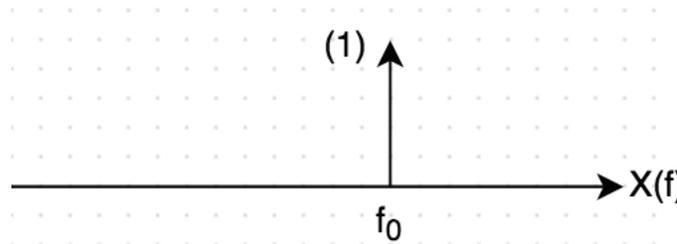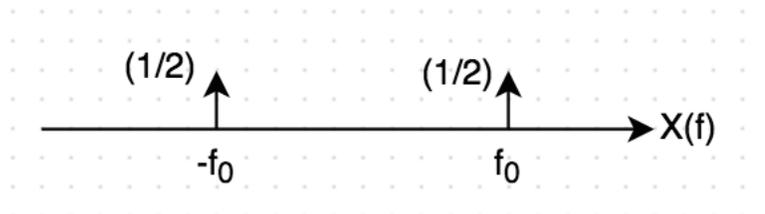

Now consider a second signal:

$$q(t) = cos(2\pi f_0 t)$$

This signal can be decomposed into

$$q(t) = \frac{1}{2}e^{i2\pi f_0 t} + \frac{1}{2}e^{-i2\pi f_0 t}$$

Here, it is clear that q(t) is really a linear combination of two separate complex exponentials, each with weight $\frac{1}{2}$. The frequency diagram for such a signal would thus look as follows:


We now work through an example problem:

$$r(t) = 2cos(200\pi t)cos(2\pi t)$$

(a) Plot $r(t)$ by hand.

(b) Break down r into its constituent frequencies.

We can approach (a) but noting that the cosine wave with the lower frequency can be used to serve as a bound as the cosine wave with higher frequency varies and is limited. This kind of pattern is used to produce a periodic beat frequency, and can be plotted as follows:
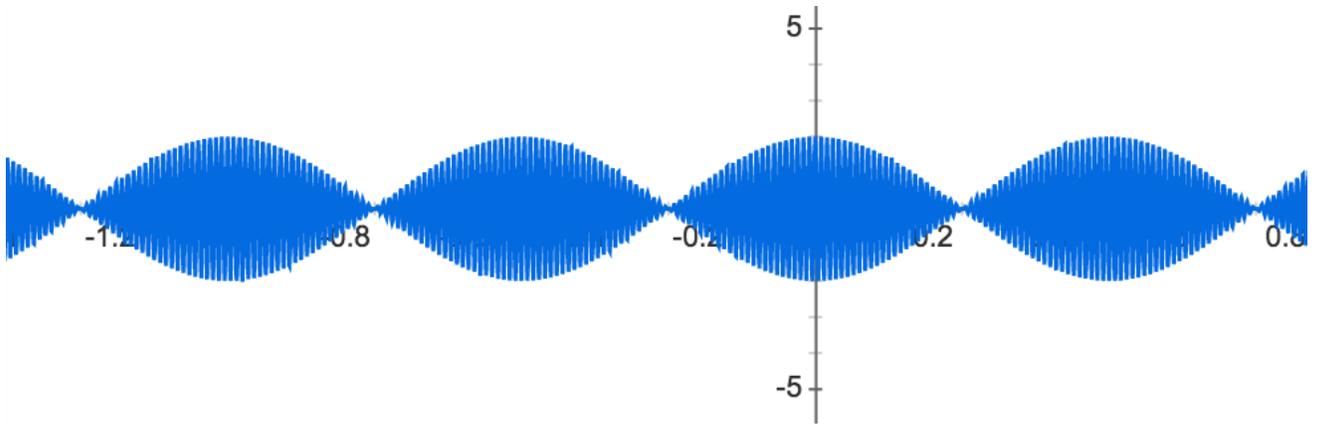
We then approach (b) by utilizing Euler's inverse formula. Substituting the corresponding values for $cos(t)$ into the given signal and simplifying, we obtain the result

$$r(t) = cos(2\pi 101t) + cos(2\pi 99t)$$

Hence we see the constituent frequencies of the given signal are $101Hz$ and $99Hz$ respectively.

Separating Signals

We now learn how to obtain independent "packets" of information transmitted by a signal. The simplest case is a signal carrying two packets of information, X_1 and X_2, each on its own separate constituent frequency f_1 and f_2 which are related to a fundamental frequency f_0 as follows:
$f_1 = k_1 f_0$ and $f_2 = k_2 f_0$ where k_1 and k_2 are integers.
Such a signal would be represented as

$$x(t) = X_1 cos(2\pi f_1 t) + X_2 cos(2\pi f_2 t)$$

To separate such a signal, we use the basic calculus method of computing the average value of a continuous function. This is done by the formula

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)dx$$

Using this, a clever trick to isolate X_1 from X_2 is to find the average value of $f(t) = x(t)cos(2\pi f_1 t)$ over one period of the function (of $T_0 = 1/f_0$). This simplifies as follows:

$$f_{avg} = \frac{1}{T_0} \int_0^{T_0} X_1 cos^2(2\pi f_1 t) + X_2 cos(2\pi f_2 t) cos(2\pi f_1 t) dt$$

$$f_{avg} = \frac{1}{T_0} \int_0^{T_0} \frac{X_1}{2} + \frac{X_1}{2} cos(4\pi f_1 t) + X_2 cos(2\pi f_1 t) cos(2\pi f_2 t) dt$$

$$f_{avg} = \frac{1}{T_0} \int_0^{T_0} \frac{X_1}{2} dt$$

$$f_{avg} = \frac{X_1}{2}$$

We can apply an analogous procedure to isolate X_2 as well. Using the same concept, we can extend this idea to more than two transmitted signals, and separate them individually. This very concept is used in many wireless transmissions such as radio to isolate the desired frequency.

Cartesian Vector Dot Products / Inner Products

As we know, the dot product of two vectors with real components is simply the sum of multiplying their corresponding components

$$\mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \sum_{k=1}^{n} x_k y_k$$

When the components of the vectors $\mathbf{x}$ and $\mathbf{y}$ are not all real, then we must use the inner product instead to compute the directions of two vectors relative to one another. The inner product is computed as

$$< \mathbf{x}, \mathbf{y}* >$$

which is simply the dot product between $\mathbf{x}$ and the complex conjugate of $\mathbf{y}$. Note that the inner product reduces to the dot product when all vector components are real.