Lecture notes by Mahesh Vashishtha (04/27/2015)

LTI Systems and the Unit Impulse Response

Consider a system H that maps any function $f : \mathbb{Z} \to \mathbb{C}$ to a single function $g : \mathbb{Z} \to \mathbb{C}$.

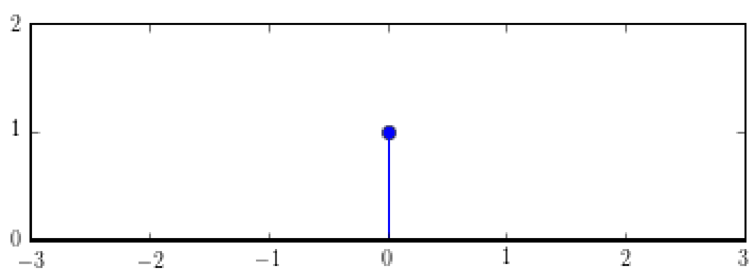
Recall that H is **linear** if, for all $x_1, x_2 : \mathbb{Z} \to \mathbb{C}$ and for all scalars α and β, $H(\alpha x_1[n] + \beta x_2[n]) = \alpha H(x_1[n]) + \beta H(x_2[n])$.
Let $y[n] = H(x[n])$. H is **time-invariant** if for all $k, n \in \mathbb{Z}$ then $y[n-k] = H(x[n-k])$

We designate H as a linear, time-invariant system, or an **LTI system**, if it is both linear and time-invariant. The rest of our discussion shall focus on the characterization of LTI systems.

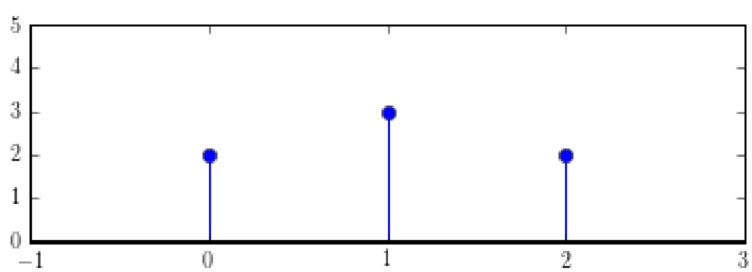
To the end of understanding the operation of a given LTI system H, it is useful to provide as input a simple signal x, and then characterize its output signal $y = H(x)$. Hence, we shall consider the application of the **unit impulse**, a signal which we shall denote as $\delta[n]$ and which is defined simply as:

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

The graph of this function is shown below:

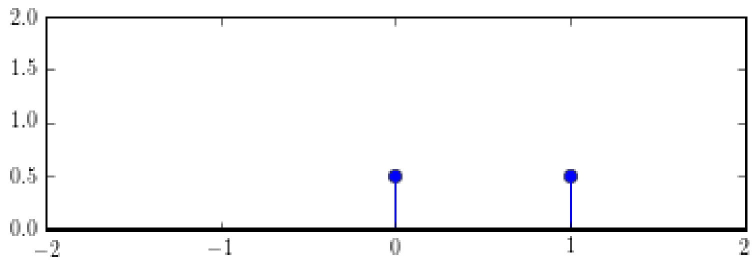


To understand the motivation for choosing this function, consider as an example the signal $x[n]$, which is 0 at all points except 0, 1, and 2, at which it takes on values of 2, 3, and 2, respectively, as shown below:

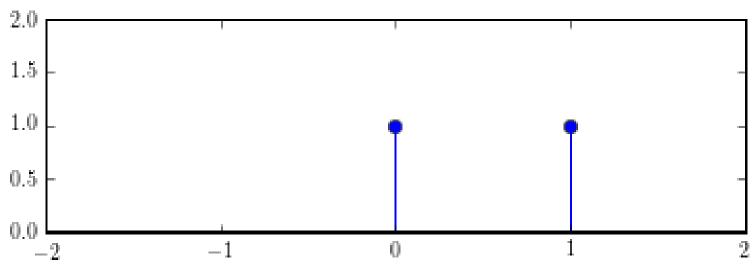


To relate $x[n]$ to $\delta[n]$, we might start by noting that $2\delta[n]$ represents the leftmost peak of $x[n]$. Moreover, we can use *shifted* versions of $\delta[n]$ to represent the other two peaks: the middle peak is $3\delta[n-1]$, and the rightmost one is $2\delta[n-2]$. Moreover, adding the scaled and shifted delta functions merely superimposes them upon each other, for each of them is nonzero at only one point. Hence we can write $x[n]$ as a *sum* of these scaled and shifted delta functions: $x[n] = 2\delta[n] + 3\delta[n-1] + 2\delta[n-2]$. We can verify from the definition of $\delta[n]$ that this will indeed produce a function that has the same value as $x[n]$ at all values n.

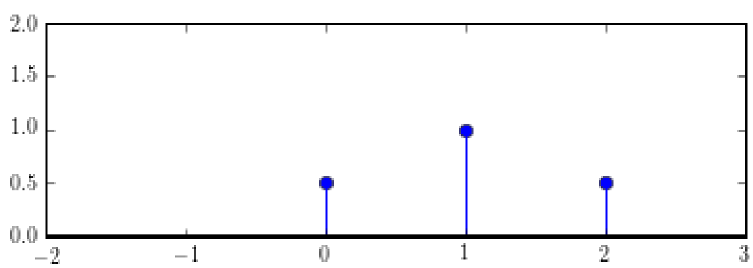
By a simple extension of the argument above, it is clear that any discrete-time signal can be written as a linear combination of $\delta[n-k]$, for various shifts k. This property is very useful in the analysis of LTI systems, because simply knowing the response of an LTI system H to $\delta[n]$ can, by linearity and time invariance, allow us to easily derive the response of H to any signal. For instance, say that we know that for some LTI system H, $H(\delta)[n]$ can be represented by the following graph:



and we want to predict $H(x)$ where $x[n]$ is as follows:



We note that the response to the peak at 0 in isolation would be the same as the response to $\delta[n]$, and that by time-invariance the response to the peak at 1 in isolation would be the same response shifted to the right. By linearity, we can superimpose these two responses to obtain the final response, which appears as:



In algebraic terms, we have used LTI system properties to show that the response to $x[n] = \delta[n] + \delta[n-1]$ is equal to $H(\delta[n]) + H(\delta[n-1])$.

Henceforth, we shall refer to the response of a system to $\delta[n]$ as the **unit impulse response**, denoted by $h[n]$. Now we can formulate the ideas above in a more compact form. Consider an arbitrary discrete-time signal $x[n]$, and call $y[n] = H(x[n])$ the response to this signal from an LTI system H. Deconstructing x into its component delta functions we can write:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

And by time invariance and linearity,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This second formulation is known as the **convolution sum**.

Complex Exponentials as Eigenfunctions

Still considering an LTI system H, we shall now apply the results of our analysis to the output $y[n]$ of the input signal $x[n] = e^{j\omega n}$. Using the convolution sum, $y[n]$ can be written as

$$\sum_{k=-\infty}^{\infty} e^{j\omega k}h[n-k]$$

But this formulation does not yield any insights, so we use the trick of rewriting the general convolution sum using the substitution $l = n-k$:

$$y[n] = \sum_{l=-\infty}^{\infty} x[n-l]h[l]$$

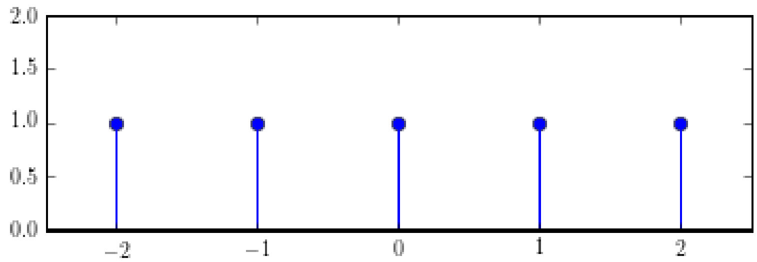
Going back to our response to $x[n] = e^{j\omega n}$, we get that

$$\begin{aligned}
y[n] &= \sum_{l=-\infty}^{\infty} e^{j\omega(n-l)}h[l] && (1)\\
&= \sum_{l=-\infty}^{\infty} (e^{j\omega n} \times e^{-j\omega l})h[l] && (2)\\
&= e^{j\omega n} \times \sum_{l=-\infty}^{\infty} e^{-j\omega l}h[l] && (3)\\
&= e^{j\omega n}H(e^{j\omega}) && (4)\\
&= H(e^{j\omega})e^{j\omega n} && (5)
\end{aligned}$$

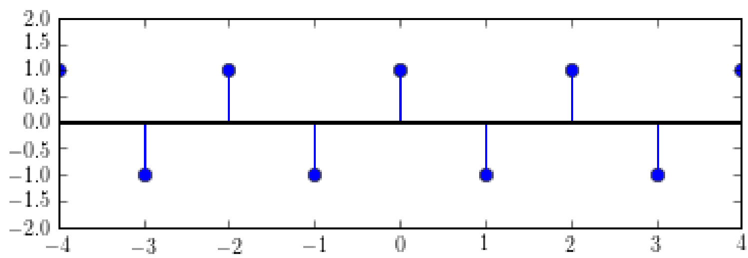
where in (4) and (5) we have denoted the infinite sum $\sum_{l=-\infty}^{\infty} e^{-j\omega l}h[l]$ as $H(e^{j\omega})$ to emphasize that it does have any dependence on n, but does depend on the frequency ω and the system H. This result shows that complex exponentials are **eigenfunctions** of LTI systems: the output signal obtained by inputting a complex exponential $e^{j\omega n}$ is a scaled version of that exponential.

Frequency Response of Complex Exponentials

Now we turn our attention to the family of complex exponentials $x[n] = e^{j\omega n}$. Consider what these functions look like for, say, $\omega = 0$:



Or for $\omega = \pi$:



These functions are actually the two extremes in terms of frequency in $\mathbb{Z}$: the first is always 1, and the second switches at every point from -1 to 1 or vice versa, being 1 at $n = 0$. It is not possible for a complex exponential in our domain to oscillate faster than $e^{j\pi n}$, since we only sample the complex exponential once every unit of time.

Now consider the LTI filter,

$$y[n] = \frac{x[n] + x[n+1]}{2}$$

If we input $x[n] = e^{j0n} = 1$, the slowest frequency, our response is

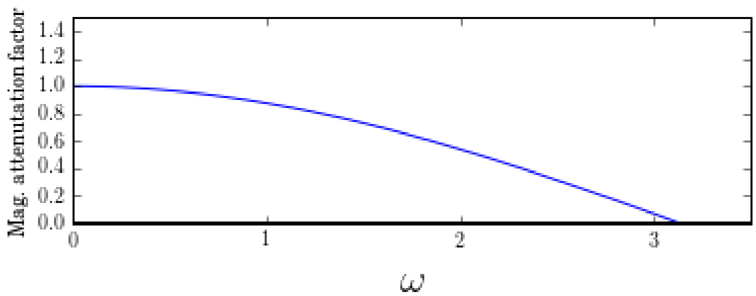
$$\frac{e^{j0n} + e^{j0(n+1)}}{2} = 1$$

In general, from the formula for $y[n]$, we see that the output resulting from input $x[n] = e^{j\omega n}$ is

$$\frac{e^{j\omega n} + e^{j\omega(n+1)}}{2} = \frac{1 + e^{j\omega}}{2} e^{j\omega n}$$

Note the connection to the fact that complex exponentials are eigenfunctions of LTI systems! What is $H(e^{j\omega})$ in this case?

Consider how $y[n]$ depends on ω, the frequency of the input oscillator. If ω is close to 0 (i.e. the input oscillation is very slow) the output signal will be only slightly attenuated, and if ω approaches π, the highest frequency for the input oscillator, the output signal approaches 0. The following graph illustrates how the magnitude of the attenuation factor $\frac{1+e^{j\omega}}{2}$ changes with ω.



This graph confirms that the two-point averaging filter is in fact a *low-pass filter*! Higher frequencies are attenuated, while lower frequencies are not.