Lecture Scribe 4/30: Quincy Huynh

1. Frequency Response of a Discrete-Time LTI System:

Diagram of system:

$$x(n) = e^{i\omega n} \longrightarrow \boxed{\mathbb{H}} \longrightarrow y(n) = \mathbb{H}(\omega)e^{i\omega n}$$

Property: The frequency response $\mathbb{H}(\omega)$ of a system is periodic with period 2π.

$$\mathbb{H}(\omega) = \sum_n h(n)e^{-i\omega n}$$
$$\mathbb{H}(\omega + 2\pi) = \mathbb{H}(\omega)$$

Proof of periodicity:

$$x(n) = e^{i(\omega+2\pi)n} \longrightarrow \boxed{\mathbb{H}} \longrightarrow y(n) = \mathbb{H}(\omega + 2\pi)e^{i(\omega+2\pi)n}$$

Consider the input $x(n) = e^{i(\omega+2\pi)n}$. Since $e^{i(\omega+2\pi)n} = e^{i\omega n} \cdot e^{i2\pi n}$ and $e^{2\pi n} = 1 \forall n \in \mathbb{Z}$, then the scalar $\mathbb{H}(\omega+2\pi) = \mathbb{H}(\omega)$ since the system is one-to-one (produces one output for one input).

Changing from radians to hertz using $\omega = 2\pi\phi$:

$$x(n) = e^{i2\pi\phi n} \longrightarrow \boxed{\widehat{\mathbb{H}}} \longrightarrow y(n) = \widehat{\mathbb{H}}(\phi)e^{i2\pi\phi n}$$

Where $\widehat{\mathbb{H}}(\phi) = \sum_n h(n)e^{-i2\pi\phi n}$ and $\widehat{\mathbb{H}}(\phi) = \mathbb{H}(\omega)\Big|_{\omega=2\pi}$.

2. Convolution

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = (x * h)(n)$$

By commutativity we can represent the impulse response as the input and the input as the impulse response, yielding:

$$h(n) \longrightarrow \boxed{x(n)} \longrightarrow y(n) = (h * x)(n)$$

3. Discrete-Time Fourier Transform

The quantity $\mathbb{X}(\omega) = \sum_n x(n)e^{-i\omega n}$ is called the Discrete Time Fourier Transform of $x(n)$. The Fourier Transform takes a signal from the time domain and returns the frequency domain representation of the signa.

Change of bases

Changing from the time domain to the frequency domain can actually be understood just as a simple change of basis! The time and frequency domain representations of a signal are simply two different representations of the same quantity. Just as one vector can have two different representations with two different bases, one signal can also have two different representations.

It turns out the basis that represents the frequency domain (i.e. the basis of complex exponentials is also orthogonal).

Consider $\mathbb{X}(\phi) = \sum_n x(n) e^{-i2\pi\phi n}$ which is the frequency domain representation of a signal. Let $\Psi_n(\phi) = e^{-i2\pi\phi n}$. Let $\widehat{X} = \sum_n x(n)\Psi_n$. Assume that $\Psi_k \perp \Psi_l$ for $k \neq l$ (We assume this for now and will prove it later.).

How do we determine $x(k)$? We project $\widehat{X}$ onto Ψ_k using the inner product.

Since $\Psi_k \perp \Psi_l$ for $k \neq l$, we know that the inner product $< \Psi_k, \Psi_l >= 0$.

$$\begin{aligned}
< \widehat{X}, \Psi_k > &= < \sum_n x(n)\Psi_n, \Psi_k > \\
&= \sum_n x(n) < \Psi_n, \Psi_k > \\
&= x(k) < \Psi_k, \Psi_k > \\
x(k) &= \frac{< \widehat{X}, \Psi_k >}{< \Psi_k, \Psi_k >}
\end{aligned}$$

Since we are in the 1-periodic space of functions:

$$\begin{aligned}
< F, G > &\triangleq \int_{<1>} F(\phi) G^*(\phi) \, \mathrm{d}\phi \\
< \Psi_k, \Psi_k > &= \int_0^1 e^{-i2\pi\phi_k} e^{i2\pi\phi_k} \, \mathrm{d}\phi \\
&= 1 \\
< \Psi_k, \Psi_l > &= \int_0^1 e^{-i2\pi\phi_k} e^{i2\pi\phi_l} \, \mathrm{d}\phi \\
&= \frac{e^{i2\pi(l-k)-1}}{i2\pi(l-k)} \\
&= \frac{0}{\text{IDK and IDC}} \text{ As long as we make sure the denominator is non-zero we are fine.} \\
&= 0
\end{aligned}$$

So the inner product gives 1 for $l - k = 0$ and 0 otherwise. This is just like the delta function!

$$< \Psi_k, \Psi_l > = \delta(l-k)$$

We use this now to compute $x(k)$ and substitute the inner product in $x(k) = \dfrac{< \widehat{X}, \Psi_k >}{< \Psi_k, \Psi_k >}$. Now we are able to go from the frequency domain back to the time domain with the general formula:

$$x(k) = \int_0^1 \widehat{X}(\phi) e^{i2\pi\phi_k} \, \mathrm{d}\phi \tag{1}$$

We can now represent the back and forth between time domain and frequency domain.

$$x(k) = \int_0^1 \widehat{X}(\phi) e^{i2\pi\phi_k} \, \mathrm{d}\phi \underset{\mathscr{F}^{-1}}{\overset{\mathscr{F}}{\rightleftarrows}} \mathbb{X}(\phi) = \sum_n x(n) e^{-i2\pi\phi n} \tag{2}$$

The above expression is true when frequency is represented in hertz. To represent frequency in radians, divide by 2π when changing from frequency to time domain because the function is 2π-periodic.

Notice that the functions of n and ϕ are sums (either by summation or integral, which is a sum of infinitesimally small deltas), and the fact that each term in the sum has some scalar multiplied by a form of Ψ_n. This means that a signal is a linear combination of "basis vectors" (here we are dealing with infinite dimensional vectors, but the basic concepts are similar to finite length vectors) in both the time and frequency domain. We've also shown that the inner product of different bases is 0, proving our assumption that the bases were orthogonal.