1. **Noise Cancelling Headphones Part 2**

Almost everyone has tried "noise cancelling" headphones at some point. The basic goal of a noise cancelling headphones is for the user to hear only the desired audio signal and not any other sounds from external sources. In order to achieve this goal, noise cancelling headphones include at least one microphone that listens to what you might have otherwise heard from external sources, and then feeds a signal in to your speakers that cancels (subtracts out) that externally-generated sound.

**Answer:** There are a lot of different solutions for this problem. This solution is aggressive and minimal, so be patient with your understanding. If your solution solves the same problem, you will receive credit.

(a) In the previous part, we had just one speaker and one microphone, but almost all headphones today have two speakers (one for each ear). Adding an extra speaker that can be driven by a separate audio stream typically makes things sound more real to us. For similar reasons, having multiple microphones to pick up ambient sounds from multiple different locations can help us do a better job of cancellation, if we can use that information in the right way.

Let’s now assume that our system has 3 microphones and 2 speakers, and that the source of our audio is stereo - i.e., we have two different audio streams $s_{left}$ and $s_{right}$ (produced by two different DACs) that represent the ideal sounds we would like the user to hear in their left and right ear. We have three microphone audio signals $s_{mic1}$, $s_{mic2}$, and $s_{mic3}$, and let’s assume that without any active noise cancellation, some fraction of the signal picked up by each microphone would be heard by the user in each of their ears. For example, $a_{1left}$ would represent the fraction of the signal picked up by microphone 1 that will be heard in the user’s left ear, $a_{2right}$ would represent the fraction of the signal picked up by
microphone 2 that will be in the user’s right ear, etc.

Still assuming no noise cancellation and assuming that the DAC/driver circuitry is ideal in producing \( s_{\text{left}} \) and \( s_{\text{right}} \), write a matrix-vector equation you could use to calculate the audio signals \( s_{\text{ear\_left}} \) and \( s_{\text{ear\_right}} \) heard by each of the user’s ears.

**Answer:**

\[
\begin{bmatrix}
  s_{\text{ear\_left}} \\
  s_{\text{ear\_right}}
\end{bmatrix}
= \begin{bmatrix}
  a_{1\text{left}} & a_{2\text{left}} & a_{3\text{left}} \\
  a_{1\text{right}} & a_{2\text{right}} & a_{3\text{right}}
\end{bmatrix}
\begin{bmatrix}
  s_{\text{mic}1} \\
  s_{\text{mic}2} \\
  s_{\text{mic}3}
\end{bmatrix}
+ \begin{bmatrix}
  s_{\text{left}} \\
  s_{\text{right}}
\end{bmatrix}
\]

(b) We define the matrix operation \( A \) to relate the signals picked up by each of the microphones to the signals heard by each ear. What matrix \( B \) should the active noise cancellation circuitry be aiming to implement in order to ensure that the user doesn’t hear any of the sounds picked up by the microphones?

**Answer:** For the setup:

\[
\begin{bmatrix}
  s_{\text{ear\_left}} \\
  s_{\text{ear\_right}}
\end{bmatrix}
= A
\begin{bmatrix}
  s_{\text{mic}1} \\
  s_{\text{mic}2} \\
  s_{\text{mic}3}
\end{bmatrix}
+ B
\begin{bmatrix}
  s_{\text{mic}1} \\
  s_{\text{mic}2} \\
  s_{\text{mic}3}
\end{bmatrix}
+ \begin{bmatrix}
  s_{\text{left}} \\
  s_{\text{right}}
\end{bmatrix}
\]

We want \( B = -A \).

(c) Using resistors and op-amps, and assuming that the microphones can be modeled as voltage sources with a source resistance of 1kΩ and whose value \( v_{\text{mic}n} \) is proportional to \( s_{\text{mic}n} \), design and sketch a circuit that would implement the cancellation matrix \( B \). You should assume that this circuit has three voltage inputs \( v_{\text{mic}1}, v_{\text{mic}2}, \) and \( v_{\text{mic}3} \) and two voltage outputs \( v_{\text{cancel\_left}} \) and \( v_{\text{cancel\_right}} \) (corresponding to the voltages that will be subtracted from the desired audio streams in order to cancel the externally-produced sounds). In order to simplify the problem, you can assume that all of the \( v_{\text{mic}} \) voltages are already centered at 0V (relative to the DAC ground).

**Answer:** Since we want to subtract \( v_{\text{cancel\_left}} \) and \( v_{\text{cancel\_right}} \) from the audio stream output, we want these values to be

\[
\begin{align*}
  v_{\text{cancel\_left}} &= a_{1\text{left}} v_{\text{mic}1} + a_{2\text{left}} v_{\text{mic}2} + a_{3\text{left}} v_{\text{mic}3} \\
  v_{\text{cancel\_right}} &= a_{1\text{right}} v_{\text{mic}1} + a_{2\text{right}} v_{\text{mic}2} + a_{3\text{right}} v_{\text{mic}3}
\end{align*}
\]

Following the design process, we can draw a block diagram for the circuit.
We can see here that the two channels are actually independent from each other. The only point they meet is the microphone voltage. Thus, we can start by designing for one channel. We want to build a circuit that adds its inputs. We have seen a circuit that does this in problem 1(c), but there we only had 2 inputs and we need 3 here. We have also seen in part (b) that we voltage divider by itself in the non-inverting input has can be used to add signals. Thus, we can start from a very simple circuit below.

\[
v_{\text{out}} = g_1 a_{1\text{left}} + g_2 a_{2\text{left}} + g_3 a_{3\text{left}} + g_2 a_{1\text{right}} + g_3 a_{2\text{right}} + g_3 a_{3\text{right}}
\]

where \( g_n = \frac{1}{R_n} \). This looks like a good start!

If we are trying to build the left cancellation voltage, then the coefficient of \( v_{\text{mic}n} \) should be \( a_{n\text{left}} \). However, if we add all the coefficients in the equation above,

\[
a_1 + a_2 + a_3 = \frac{g_1}{g_1 + g_2 + g_3} + \frac{g_2}{g_1 + g_2 + g_3} + \frac{g_3}{g_1 + g_2 + g_3} = \frac{g_1 + g_2 + g_3}{g_1 + g_2 + g_3} = 1
\]

Thus this circuit has a constraint on the values of \( a \) we can choose - not good! We need to think of a tweak that lets us choose \( a \) arbitrarily. Here’s an idea: what if we put a dummy input that is tied to ground? We saw that the total of the gains must equal to 1, so if we have a dummy input we can set all of the gains to be anything as long as the total is less than 1. We have also seen that whenever we have a voltage divider we are only dealing with ratios of the resistances. Thus, we can put the three resistors we have right now in terms of our new resistor.
Let's rewrite the KCL at the output node.

\[
\frac{v_{\text{mic}1} - v_{\text{out}}}{\alpha R} + \frac{v_{\text{mic}2} - v_{\text{out}}}{\beta R} + \frac{v_{\text{mic}3} - v_{\text{out}}}{\gamma R} + \frac{0 - v_{\text{out}}}{R} = 0
\]

\[
v_{\text{out}} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 1 \right) = \frac{v_{\text{mic}1}}{\alpha} + \frac{v_{\text{mic}2}}{\beta} + \frac{v_{\text{mic}3}}{\gamma}
\]

This looks good... but now we can only make the circuit work if the total of the \( a \)'s is less than 1. If we want to pick arbitrary values for \( a \), we cannot have this constraint. How can we achieve this? The problem is actually simpler than it seems! We have a controllable gain on each of the microphone voltages and we have added them together, we just need to amplify the sum. We can use a non-inverting amplifier to do this.

The circuit we have right now amplifies our last \( v_{\text{out}} \) by \( \frac{R+1+R_2}{R_2} \). But what do we want this gain to be? Our assumption is that the \( a \)'s are fractions between 0 and 1, which means the sum of \( a \)'s is 3. We have also built a circuit that works if the sum of \( a \)'s is below 1. Thus, we need a gain of 3 to cover all possible values for \( a \). Using the gain formula, we can pick \( R_1 = 2\Omega \) and \( R_2 = 1\Omega \). With the amplifier, then our output voltage is

\[
v_{\text{out}} = \frac{3v_{\text{mic}1}}{g\alpha + g\beta + g\gamma + 1} + \frac{3v_{\text{mic}2}}{g\alpha + g\beta + g\gamma + 1} + \frac{3v_{\text{mic}3}}{g\alpha + g\beta + g\gamma + 1}
\]
If we want $V_{out} = a_1 v_{mic1} + a_2 v_{mic2} + a_3 v_{mic3}$, then

$$\frac{3}{\alpha \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 1 \right)} = a_1$$

$$\frac{3}{\beta \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 1 \right)} = a_2$$

$$\frac{3}{\gamma \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 1 \right)} = a_3$$

If we divide the first equation by the second, we get

$$\frac{\beta}{\alpha} = \frac{a_1}{a_2} \quad \alpha = \frac{a_2}{a_1} \beta$$

Similarly with the second and third equations,

$$\frac{\gamma}{\beta} = \frac{a_2}{a_3} \quad \beta = \frac{a_3}{a_2} \gamma \quad \text{so} \quad \alpha = \frac{a_3}{a_1} \gamma$$

Replacing $\alpha$ and $\beta$ in the third equation,

$$3 = a_3 \gamma \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + 1 \right)$$

$$3 = a_3 \left( \frac{a_1}{a_3} + \frac{a_2}{a_3} + 1 + \gamma \right)$$

$$\gamma = \frac{3 - a_1 - a_2 - a_3}{a_3}$$

$$\alpha = \frac{3 - a_1 - a_2 - a_3}{a_1}$$

$$\beta = \frac{3 - a_1 - a_2 - a_3}{a_2}$$

Whew. Almost there. Remember that our microphone voltages have source resistances of $1k\Omega$. We can treat this as a series resistance, as part of the resistors $\alpha R$, $\beta R$ and $\gamma R$. However, we need 2 channels - one for each ear. When we hook a second copy of the circuit up, the node we connect the second circuit acts as another voltage divider.

![Diagram of circuit](image)
This means if the left circuit node in the circuit above is pulled down, there it can affect the right circuit node by pulling down the voltage divider. Thus, we cannot just hook up the second circuit into the first. The simplest way to solve this is to use a unity-gain buffer. A buffer would isolate the source resistance from the actual circuit so we can just connect the circuits for the two channels in parallel and would output the actual microphone voltage since there is no current coming into the op-amp input terminals. For each microphone, we need the buffer below.

Now that we have all the building blocks we need, we can construct the two-channel noise cancelling circuit. \( v_{\text{mic} n}' \) is connected to the output of the microphone buffers. We can choose an arbitrary value for \( R \), for example 1 k\( \Omega \).

\[
\begin{align*}
\alpha_1 &= \frac{3}{a_{1\text{left}}} - \frac{a_{2\text{left}} - a_{3\text{left}}}{a_{1\text{left}}} \\
\beta_1 &= \frac{3}{a_{1\text{left}}} - \frac{a_{2\text{left}} - a_{3\text{left}}}{a_{2\text{left}}} 
\end{align*}
\]
\[
\gamma_1 = \frac{3 - a_{1left} - a_{2left} - a_{3left}}{a_{3left}}
\]
\[
\alpha_2 = \frac{3 - a_{1right} - a_{2right} - a_{3right}}{a_{1left}}
\]
\[
\beta_2 = \frac{3 - a_{1right} - a_{2right} - a_{3right}}{a_{2right}}
\]
\[
\gamma_2 = \frac{3 - a_{1right} - a_{2right} - a_{3right}}{a_{3right}}
\]

(d) **BONUS**: Building upon your solutions to all previous parts, and otherwise making the same assumptions about the relative voltage ranges of \(v_{mic1}\), \(v_{mic2}\), and \(v_{mic3}\) and available supply voltages, sketch the complete circuit you would use to create the stereo audio on the two speakers while cancelling the noise picked up by the three microphones.

**Answer:** We already have a circuit that does subtraction from part (b) and a circuit that computes the noise cancelling signal in part (e). We just have to combine the two circuits such that it implements the matrix \(B\) in part (d). A naive way to do this is to just chain them up together, but we will try to be more aggressive and try to only use 1 op-amp for each channel, on top of the microphone buffers. The block diagram for this circuit is as below.

Going back to part (b), we can see that to add something we have to feed it into the positive input and to subtract something we have to feed it into the negative input of the op-amp. We can take this principle and combine the circuits in part (b) and (e). Since we still need to shift the DAC voltages but do not need to shift the microphone voltages, we can try the circuit below. As we have seen previously, when we have multiple channels the source resistance becomes a problem. Thus we can use the same technique as before by using a buffer.
If we take a closer look at the boxed part of the circuit above, we notice that this is actually one of the circuits we experimented in part (e)! The only difference is in part (e) we wanted to add the inputs so we connected it to the non-inverting input of the op-amp while here we want to subtract the signals so we connect it to the inverting input of the op-amp.

Recall that the $v_{in}$ range is $-0.375V$ to $0.375V$, and it has to be amplified 4 times. We can write the KCL equation in the inverting input of the op-amp.

\[
\frac{v_{mic1} - v_{in}}{\alpha} + \frac{v_{mic2} - v_{in}}{\beta} + \frac{v_{mic3} - v_{in}}{\gamma} + \frac{v_{out} - v_{in}}{R_1} + \frac{0 - v_{in}}{R_2} = 0
\]

\[
\frac{v_{out}}{R_1} = v_{in} \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) - \frac{v_{mic1}}{\alpha} - \frac{v_{mic2}}{\beta} - \frac{v_{mic3}}{\gamma}
\]

Just as before, we can compare this formula to the output we want. In this case, we want $v_{out} = 4v_{in} - a_1 v_{mic1} - a_2 v_{mic2} - a_3 v_{mic3}$. Thus,

\[
\frac{R_1}{\alpha} + \frac{R_1}{\beta} + \frac{R_1}{\gamma} + 1 + \frac{R_1}{R_2} = 4 \quad \frac{R_1}{\alpha} = a_1 \quad \frac{R_1}{\beta} = a_2 \quad \frac{R_1}{\gamma} = a_3
\]

From the first equation,

\[
a_1 + a_2 + a_3 + 1 + \frac{R_1}{R_2} = 4
\]
Thus, if we pick a value for $R_1$, we can use the formulas above to calculate $\alpha$, $\beta$, $\gamma$ and $R_2$.

Now that we have a working circuit for one speaker, we can duplicate this circuit to have two speakers. Notice that in the circuit below we can use the same value for $R_1$ in the two channels, but we have to keep $R_2$ as a variable (hence it is replaced with $R_3$ in the right channel). This is because $R_1$ is a free variable. If we choose a value for $R_1$ arbitrarily, we can calculate what the other resistor values have to be with the equations we have derived.

We have seen that if we choose values for $R_1$ and $R_3$ arbitrarily, we can find the other resistor values.

\[
\begin{align*}
\alpha_1 &= \frac{R_1}{a_{1\text{left}}} & \beta_1 &= \frac{R_1}{a_{2\text{left}}} & \gamma_1 &= \frac{R_1}{a_{3\text{left}}} & R_2 &= \frac{R_1}{3 - a_{1\text{left}} - a_{2\text{left}} - a_{3\text{left}}} \\
\alpha_2 &= \frac{R_1}{a_{1\text{right}}} & \beta_2 &= \frac{R_1}{a_{2\text{right}}} & \gamma_2 &= \frac{R_1}{a_{3\text{right}}} & R_3 &= \frac{R_1}{3 - a_{1\text{right}} - a_{2\text{right}} - a_{3\text{right}}} 
\end{align*}
\]