EECS 16A Designing Information Devices and Systems I
Spring 2016 Elad Alon, Babak Ayazifar Discussion 3A

Reference Definitions

Vector spaces: A vector space V is a set of elements that is closed under vector addition and scalar multiplication. For V to be a vector space, the following conditions must hold for every $\vec{u}, \vec{v}, \vec{z} \in V$ and for every $c, d \in \mathbb{R}$

No escape property (addition) $\vec{u} + \vec{v} \in V$,

No escape property (scalar multiplication) $c\vec{u} \in V$,

Commutativity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,

Associativity of vector addition $(\vec{u} + \vec{v}) + \vec{z} = \vec{u} + (\vec{v} + \vec{z})$,

Additive identity There is $\vec{0} \in V$ such that for all $\vec{u}$, $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$,

Existence of inverse For every $\vec{u}$, there is element $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = 0$,

Associativity of scalar multiplication $c(d(\vec{u})) = (cd)\vec{u}$,

Distributivity of scalar sums $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,

Distributivity of vector sums $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,

Scalar multiplication identity There is $1\vec{u} = \vec{u}$.

The most important of the above properties are the *no escape properties*. These demonstrate that the vector space is closed under addition and scalar multiplication. That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the following two conditions hold for any two vectors $\vec{u}, \vec{v} \in W$, and any scalar $c \in \mathbb{R}$:

No escape property (addition) $\vec{u} + \vec{v} \in W$

No escape property (scalar multiplication) $c\vec{u} \in W$

Note that these are the only properties we need to establish to show that a subset of a vector space is a subspace! The other properties of the underlying vector space come for free, so to speak.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$, as well as their subspaces.

Basis: A *basis* for a vector space is a *set of linearly independent vectors* that *spans the vector space*.

So, if we want to check whether a set of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ forms a basis for a vector space V, we check for two important properties:

(a) $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are linearly independent.

(b) $span(\vec{v}_1, \vec{v}_2, ..., \vec{v}_k) = V$

(Write standard basis as standard example of a basis?)

As we move along, we'll learn how to identify and/or construct a basis, and we'll also learn some interesting properties of bases.

1. **Lecture Review 2/9/16**

2. **Identifying a subspace: Proof exercise!**

   Is the set

$$V = \left\{ \vec{v} : c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ where } c, d \in \mathbb{R} \right\}$$

   a subspace of $\mathbb{R}^3$? Why/why not?

3. **Identifying a basis**

   Does each of these sets describe a basis of some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4. **Constructing a basis**

   Let's consider a subspace of $\mathbb{R}^3$, V, that has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, we have $a_1 = 2(a_2 + a_3)$.

   Find a basis for V. What is the dimension of V?

5. **Exploring dimensionality, linear independence and bases**

   In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimensionality of a vector space/subspace, and basis.

   Let's consider the vector space $\mathbb{R}^m$, and a set of n vectors $\vec{v}_1, \vec{v}_2, ...\vec{v}_n$ in $\mathbb{R}^m$.

   (a) For the first part of the problem, let $m > n$. Can $\vec{v}_1, \vec{v}_2, ...\vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

   (b) Let $m = n$. Can $\vec{v}_1, \vec{v}_2, ...\vec{v}_n$ form a basis of $\mathbb{R}^m$? Why/why not? What conditions would we need?

(c) Now, let $m < n$. Can $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ form a basis of $\mathbb{R}^m$? What vector space could they form a basis for? (Hint: think about whether the vectors can now be linearly independent.)

6. Are Some Bases Better Than Others?

In general there can be many bases for the same vector space. To see this, let's consider the vector space $\mathbb{R}^3$. Clearly,

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. This is called the *standard basis*.

(a) Show that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^3$.

(b) Show that $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^3$.

(c) Which of the two bases might you prefer to use to describe $\mathbb{R}^3$? Why?