1. Unit Spheres

The unit sphere of a given norm $\| \cdot \|$ is the set of vectors $x$ for which $\| x \| = 1$. For the following norms (in $\mathbb{R}^2$), plot their unit sphere:

(a) $\| \cdot \|_1$
(b) $\| \cdot \|_2$
(c) $\| \cdot \|_\infty$

**Solution:**

$\| \cdot \|_1$ is the sum of the absolute value of the elements (also known as "Manhattan distance"). The corresponding unit sphere is a rotated square.

$\| \cdot \|_2$ is standard Euclidean distance, so the unit sphere follows our standard idea of the set of points equidistant from the origin: a circle.

$\| \cdot \|_\infty$ is the maximum of the absolute value of the elements. The unit sphere is a square. The three unit spheres are shown below (source: Wikipedia)
2. Matrix Inner Products

First, a definition: the trace of a square matrix \( A \), denoted \( \text{tr}(A) \), is the sum of its diagonal entries.

An inner product for matrices \( A, B \in \mathbb{R}^{m \times n} \) is:

\[
\langle A, B \rangle = \text{tr}(A^T B)
\]

(a) Let \( a_i \) be the \( i^{\text{th}} \) column of \( A \) and \( b_i \) be the columns of \( B \). Show that:

\[
\text{tr}(A^T B) = \sum_{i=1}^{n} \langle a_i, b_i \rangle
\]

**Solution:**

Break the multiplication \( A^T B \) down into columns.

\[
A^T B = \begin{bmatrix}
\bar{a}_1^T \\
\bar{a}_2^T \\
\vdots \\
\bar{a}_n^T
\end{bmatrix} \begin{bmatrix}
\bar{b}_1 \\
\bar{b}_2 \\
\vdots \\
\bar{b}_n
\end{bmatrix}
\]

Considering each element, \( (A^T B)_{ij} = \bar{a}_i^T \bar{b}_j = \langle \bar{a}_i, \bar{b}_j \rangle \).

The trace is the sum of the diagonals:

\[
\text{tr}(A^T B) = \sum_{i=1}^{n} (A^T B)_{ii} = \sum_{i=1}^{n} \langle a_i, b_i \rangle
\]

(b) Confirm that \( \langle A, B \rangle \) is in fact an inner product by showing the following:

i. \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \)

ii. \( \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \)

iii. \( \langle u, v \rangle = \langle v, u \rangle \)

iv. \( \langle u, u \rangle \geq 0 \) with equality if and only if \( u = 0 \).

**Solution:**

Each of the transformations directly affects the columns of \( A \) or \( B \), then starting with the sum definition of \( \text{tr}(A^T B) \) given above, expand using linearity and inner product properties of \( \langle a_i, b_i \rangle \).
3. Row Space

Consider:

\[ V = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix} \]  
(1)

Row reducing this matrix yields:

\[ U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  
(2)

(a) Show that the row space of U and V are the same. Argue that in general, Gaussian elimination preserves the row space.

**Solution:**
To show that the span of sets of vectors are the same, just show that each vector in one set can be written in terms of vectors in the other. Because of the structure of U here, this is easy. Eg. if the rows of U are \( u_i^T \) and the rows of V are \( v_i^T \), then \( v_1 = 2u_1 + 4u_2 \) and so on, easily found because the first two elements in \( v_i \) are the weights for \( u_i \). From this we see \( v_2 = 4u_1, v_3 = 6u_1 + 4u_2, v_4 = -2u_1 + 4u_2 \).

In general: the valid Gaussian elimination operations are to scale rows, swap rows, or add rows to each other. Since the span of a set of vectors is all linear combinations of the vectors, it is not affected by scaling, ordering, or summing.

(b) Show that the null space of U and V are the same. Argue generally that Gaussian elimination preserves the null space.

**Solution:**
To show the null spaces are the same, simply compute the null space and check. The null space for both is span\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \}.

In general: if there is a vector \( x \) in the null space of \( V \), then \( Vx = 0 \). One interpretation is that for every row \( v_i^T \), \( v_i^T x = 0 \). During Gaussian elimination we scale/swap/add rows to each other and all of them satisfy \( v_i^T x = 0 \). Therefore all the rows always continue to satisfy the relation during GE.