1. Mechanical Problems

(a) Compute the determinant of \[
\begin{bmatrix}
  2 & 0 \\
  0 & 3
\end{bmatrix}
\]

**Solution:**
We can use the form of a 2x2 determinant from lecture:

\[
det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc
\]

So:

\[
det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 = 6
\]

(b) Compute the determinant of \[
\begin{bmatrix}
  2 & 1 \\
  0 & 3
\end{bmatrix}
\]

**Solution:**

\[
det\left(\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 1 \cdot 0 = 6
\]

(c) Compute the determinant of

\[
\begin{bmatrix}
  -4 & 0 & 0 & 0 \\
  0 & 17 & 0 & 0 \\
  0 & 0 & -31 & 0 \\
  0 & 0 & 0 & 2
\end{bmatrix}
\]

**Solution:**
This is a diagonal matrix. So what it does is to scale each axis by the appropriate term on the diagonal. Consequently, since for us the determinant is defined to be the oriented hypervolume of the unit hypercube as transformed by the matrix, we immediately know that the determinant is 4216 since that is the product of the diagonal entries. The negative signs here represent flips — reflections about that particular axis. These should flip the sign of the oriented hypervolume (this is what makes it oriented). However, the two flips cancel.

Alternatively, we can use Gaussian elimination (which if you remember from lecture, was justified by the more fundamental volume-based definition of the determinant). We use Gaussian elimination on the matrix to reduce it to the identity, keeping track of all changes made:

i. Multiply R1 by \(-\frac{1}{4}\). Denote \(c_1 = -\frac{1}{4}\) for bookkeeping.

\[
A = \begin{bmatrix}
  -4 & 0 & 0 & 0 \\
  0 & 17 & 0 & 0 \\
  0 & 0 & -31 & 0 \\
  0 & 0 & 0 & 2
\end{bmatrix} \sim \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 17 & 0 & 0 \\
  0 & 0 & -31 & 0 \\
  0 & 0 & 0 & 2
\end{bmatrix}
\]
ii. Multiply R2 by \( \frac{1}{17} \). Denote \( c_2 = \frac{1}{17} \) for bookkeeping.

\[
A_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 17 & 0 & 0 \\
0 & 0 & -31 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

iii. Multiply R3 by \( -\frac{1}{31} \). Denote \( c_3 = -\frac{1}{31} \) for bookkeeping.

\[
A_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -31 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]

iv. Multiply R4 by \( \frac{1}{2} \). Denote \( c_4 = \frac{1}{2} \) for bookkeeping.

\[
A_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

v. The determinant of this last matrix is known to be 1. Now we trace our steps, knowing that scaling a row by a constant \( c_i \) scales the determinant by \( c_i \). Suppose the initial matrix is \( A \). Then, the following must be true:

\[
det(I) = c_4 \cdot det(A_4) = c_4 \cdot c_3 \cdot det(A_3) = c_4 \cdot c_2 \cdot c_1 \cdot det(A_2) = c_4 \cdot c_3 \cdot c_2 \cdot c_1 \cdot det(A)
\]

We know \( det(I) = 1 \), so we substitute and solve for \( det(A) \):

\[
det(I) = c_4 \cdot c_3 \cdot c_2 \cdot c_1 \cdot det(A) = \frac{1}{2} \cdot \frac{1}{17} \cdot \frac{1}{31} \cdot \frac{1}{2} \cdot det(A) = \frac{1}{2184} \cdot det(A)
\]

\[
det(A) = 4216 \cdot det(I) = 4216 \cdot 1 = 4216
\]

so the final answer is 4216. Notice that this is exactly the product of the diagonal entries, which is what we saw from the definition must be true for matrices that have zeros everywhere except on the main diagonal.

2. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by \( a \) will increase the determinant by \( a \), and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by \( -1 \). The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zeros below its diagonal. For example a \( 3 \times 3 \) upper triangular matrix is:

\[
\begin{bmatrix}
a_1 & a_2 & a_3 \\
0 & b_2 & b_3 \\
0 & 0 & c_3
\end{bmatrix}
\]

By considering row operations and what they do to a determinant, argue that the determinant of a general \( n \times n \) upper-triangular matrix is the product of its diagonal entries, if they are non-zero. For example, the determinant of the \( 3 \times 3 \) matrix above is \( a_1 \times b_2 \times c_3 \) if \( a_1, b_2, c_3 \neq 0 \).
Solution:  A $n \times n$ upper-triangular matrix looks like:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix}$$

For every row $i$, divide it by $a_{i,i}$. Then we get 1s on the diagonal.

$$A' = \begin{bmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \cdots & \frac{a_{1,n}}{a_{1,1}} \\ 0 & 1 & \cdots & \frac{a_{2,n}}{a_{2,2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The determinant of this new matrix is reduced by $1/(a_{1,1} \times \cdots \times a_{n,n})$ times:

$$\det A' = \frac{\det A}{a_{1,1} \times \cdots \times a_{n,n}}$$

Finally, starting from the last row, subtract multiples of the row from the ones above it, so that we get the $n \times n$ identity matrix $I_n$. This does not change the determinant since we are subtracting rows from each other. Thus:

$$1 = \det I_n = \det A' = \frac{\det A}{a_{1,1} \times \cdots \times a_{n,n}}$$

$$\det A = a_{1,1} \times \cdots \times a_{n,n}$$

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

Solution:  If an upper-triangular matrix has a zero in its diagonal, it cannot be row-reduced into an identity matrix, this means that that its rows are linearly dependent. Therefore its determinant is zero, which is the product of all diagonal entries (since one of them is zero).
3. Steady State Reservoir Levels We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure ??.

![Figure 1: Reservoir pumps system](image)

(a) Write the transition matrix representing the pumps system in the problem.

Solution:

\[
T = \begin{bmatrix}
0.2 & 0.5 & 0.4 \\
0.4 & 0.3 & 0.3 \\
0.4 & 0.2 & 0.3 \\
\end{bmatrix}
\]

(b) Assuming you start the pumps with water levels \( A_0 = 129, B_0 = 109, C_0 = 0 \) (in kiloliters). What would be the steady state water levels (in kiloliters) according to the pumps system described in the problem?

Hint: If \( \vec{x} \) is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with \( A_{ss}, B_{ss} \) and \( C_{ss} \) kiloliters of water, respectively and apply the pumps once?

Hint II: Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.

Solution: If \( \vec{x}_{ss} \) is a vector describing the steady state levels of water in the reservoirs then we know that \( T \cdot \vec{x}_{ss} = \vec{1} \cdot \vec{x}_{ss} \) — that is, applying the pumps one more time wouldn’t change the level of water in any of the reservoirs. This means that \( \vec{x}_{ss} \) is an eigenvector of \( T \), associated with the eigenvalue \( \lambda = 1 \). Therefore,

\[
\vec{x}_{ss} \in Null (T - \vec{1} \cdot I) = Null \left( \begin{bmatrix}
0.2 & 0.5 & 0.4 \\
0.4 & 0.3 & 0.3 \\
0.4 & 0.2 & 0.3 \\
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \right) = Null \left( \begin{bmatrix}
-0.8 & 0.5 & 0.4 \\
0.4 & -0.7 & 0.3 \\
0.4 & 0.2 & -0.7 \\
\end{bmatrix} \right)
\]
We calculate the nullspace of the matrix 
\[
\begin{bmatrix}
-0.8 & 0.5 & 0.4 \\
0.4 & -0.7 & 0.3 \\
0.4 & 0.2 & -0.7
\end{bmatrix}
\]
which is simply \( \text{span} \left\{ \begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix} \right\} \).

Which means that our steady state reservoirs levels vector is of the form: 
\[
\begin{bmatrix} 43\alpha \\ 40\alpha \\ 36\alpha \end{bmatrix}
\]

That is, no water is lost by running the pumps system. So we know that the total amount of water in the reservoirs at any point in time will be \( 129 + 109 + 0 = 238 \) (equal to the original total amount of water in the system). Therefore, we are looking for an eigenvector that its components sum to 238. In other words, we are looking for \( \alpha \) such that \( 43\alpha + 40\alpha + 36\alpha = 238 \) which yields \( \alpha = 2 \). Therefore, the steady state levels of water in the reservoirs will be 
\[
\begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}
\]