1. Can You Hear the Shape of a Drum?

This problem is inspired by a popular problem posed by Mark Kac in his article "Can you hear the shape of a drum?" Kac’s question was about different shapes of drums. Here’s what he wanted to know: if the shape of a drum defines the sound that’s made when we strike it, can we listen to the drum and automatically infer its shape? Deep down, this is really a question about eigenvalues and eigenvectors of a matrix. The vibrational dynamics of a particularly shaped drum membrane can be captured by a system of linear equations represented by a matrix. The eigenvalues and eigenvectors of this matrix reveal interesting properties about the drum that will help us answer the question: can we hear its shape?

We’ll use a model of vibration given by the equation,

\[ \nabla^2 u(x,y) + \lambda u(x,y) = 0 \]

Where \( u \) is the amount of displacement of the drum membrane at a particular location \( (x,y) \), and \( \lambda \) is a parameter (which will turn out to be an eigenvalue, as you will see). The “\( \nabla^2 \)” is an operator called the “Laplacian,” and just stands for taking the 2nd \( x \)-partial-derivative and adding it to the 2nd \( y \)-partial-derivative:

\[ \nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x,y-h) + u(x-h,y)}{h^2} \]

I’ve given you an approximation for the Laplacian above, which is the key to formulating this problem as a matrix equation. This equation is known as the “5-point finite difference equation” because it uses five points.

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1Marc Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966), 1-23.
points (the point at \( x, y \) and each of its nearest neighbors) to approximate the value of the Laplacian. The last thing you’ll need before we start is the 1D version of this equation, to start:

\[
\frac{d^2u}{dx^2} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}
\]

(Note: for 1D the Laplacian simplifies to a regular 2nd derivative; the factor on the \( u(x) \) is 2 instead of 4; and there are only 3 points!)

(a) First we’ll do a simple model: a violin string. Write the finite difference matrix problem for a 1×5 1D violin string as shown in Figure 1. Use the model shown above to derive your matrix. You can make the assumption that the ends of the string (points 0 and 4) are anchored, so they always have a displacement of zero. Assume that the length of the string is 1 meter (even though that’s kind of long for a violin...) (Note: there are only 3 unknowns here!)

![Figure 1: A 5-point model of a violin string.](image)

**Solution:**

\[
\begin{bmatrix}
-\lambda & u[1] \\
\end{bmatrix}
\begin{bmatrix}
u[1] \\
u[2] \\
u[3]
\end{bmatrix}
\]

(b) For our vibrating string, find the 3 eigenvalues (\( \lambda \)) of the matrix \( A \).

**Solution:**

\[
\lambda_1 = \sqrt{\frac{2}{0.25^2}} - 2 = -9.37...
\]

\[
\lambda_2 = \frac{-2}{0.25^2} = -32
\]

\[
\lambda_3 = -\sqrt{\frac{2}{0.25^2}} - 2 = -54.6...
\]

These can be found by solving the characteristic polynomial for the matrix \( A \) given in part (a):

\[
A = \begin{bmatrix}
-32 & 16 & 0 \\
16 & -32 & 16 \\
0 & 16 & -32
\end{bmatrix}
\]

\[
A\vec{x} = \lambda\vec{x}
\]

\[
(A - \lambda I)\vec{x} = 0
\]

\[
\det (A - \lambda I) = \begin{vmatrix}
-32 - \lambda & 16 & 0 \\
16 & -32 - \lambda & 16 \\
0 & 16 & -32 - \lambda
\end{vmatrix} = 0
\]
\[
\det (A - \lambda I) = (-32 - \lambda) \begin{vmatrix} 16 & 16 & 16 \\ -32 - \lambda & -16 & 0 \\ -32 - \lambda & 16 & 0 \end{vmatrix} + (0) \begin{vmatrix} -32 - \lambda & 16 \\ 16 & 16 \end{vmatrix} = 0
\]

We can immediately read off \( \lambda = -32 \) as one solution.

Next, we solve the quadratic:

\[
(-32 - \lambda)^2 - 2(16^2) = 0
\]

\[
(-32 - \lambda)^2 = 2(16^2)
\]

\[
\sqrt{(-32 - \lambda)^2} = \sqrt{2(16^2)}
\]

This yields two more eigenvalue solutions: First,

\[
(-32 - \lambda) = 16\sqrt{2}
\]

\[
\lambda = \frac{-2\sqrt{2} - 2}{0.25^2}
\]

Second,

\[
-(32 - \lambda) = 16\sqrt{2}
\]

\[
\lambda = \frac{\sqrt{2} - 2}{0.25^2}
\]

(c) For the vibrating string, find the 3 eigenvectors \( \vec{u} \) that correspond to the \( \lambda \)'s from part b. What do these vectors look like?

**Solution:**

\[
\vec{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix}
\]

\[
\vec{u}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}
\]

\[
\vec{u}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix}
\]

Should be sinusoidal, with zeros at boundaries.

(d) What do you think the eigenvalues mean for our vibrating string? (Hint: what does a larger eigenvalue seem to indicate about the corresponding eigenvector?)

**Solution:** As the eigenvalue gets larger, the string should be more "wiggly." This corresponds to a vibration at a higher frequency (also called pitch). Out of the scope of this class, the eigenvalue also corresponds to the "energy" in the mode. You would need information about the density of the string and an air damping constant to derive the sound (or pitch) generated by the vibration of the string.
Using what you know from part (a) of this problem, we will write down the 5-point finite difference equation for a 5×5 square drum in the form of a matrix problem so that it has the same form as

\[-\lambda \ddot{u} = A \ddot{u}\]

In this formulation, as in the 1D formulation, each row of \( A \) will correspond to the equation of motion for one point on the model. In our 5×5 grid, we will be modeling the motion of the inner 3×3 grid, since we will assume the membrane is fixed on the outer border. Since there are 9 points that we are modeling, this corresponds to 9 equations and 9 unknowns, so \( A \) should be 9×9.

![Figure 2: A 25-point model of a drum membrane.](image)

(e) Based on our intuition from the 1D problem, what do the eigenvalues and eigenvectors correspond to in the 2D problem?

**Solution:** Eigenvectors represent a 2D function which is a possible standing wave for the particular shape of drum that we have. Eigenvalues correspond to the frequency-squared of the corresponding eigenvector.

(f) Write down the 9×9 matrix, \( A \), for the drum in Figure 2. It should have some symmetry, but be careful with the diagonals.

**Solution:**

\[
A = \frac{1}{h^2} \begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{bmatrix}
\]

Note the missing zeros on the +1 and -1 diagonals. Since the side length of the square mesh is not given, the spacing, \( h \), could be chosen arbitrarily by the student. If \( h = 0.25 \) is used (as in previous parts), the -4’s will change to -64’s, and 1’s will change to 16’s. Eigenvalues will scale with \( \frac{1}{h^2} \) as well.
(g) In the IPython Notebook, implement a function to solve the finite difference problem for a square drum of any side-length (though keep the side-length short at first, so that you don’t run into memory problems!). What are the eigenvalues of the $5 \times 5$ drum?

**Solution:** See IPython solution notebook.

(h) Using some of the built-in functionality, you can construct a drum with any polygonal shape. There are two shapes already implemented, with the shapes shown below. The code already included will construct the $A$ matrix given a polygon and a grid. Find the first 10 vibrational modes of each drum, and the associated eigenvalues (this is analogous to finding the first 10 eigenvectors of each $A$ matrix, and the associated eigenvalues). Plot the 0th, 4th, and 8th modes using a contour plot.

(i) These two drums are different shapes. Do they sound the same? Why or why not? Can you hear the shape of a drum?

**Solution:** The drums given in the notebook sound the same. We know this because the eigenvalues of each drum are identical (to numerical precision implemented by your Python installation). Therefore, you cannot hear the shape of a drum!

2. Traffic Flows

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this “flow conservation” to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem we will explore this concept.

(a) Let’s begin with a network with three intersections, $A$, $B$ and $C$. Define the flows $t_1$ as the rate of cars (cars/hour) on the road between $B$ and $A$, $t_2$ as the rate on the road between $C$ and $B$ and $t_3$ as the rate on the road between $C$ and $A$.

![Figure 3: A simple road network.](image)

(Note: The directions of the arrows in the figure are only the way that we define the flow by convention. If there were 100 cars per hour traveling from $A$ to $C$, then $t_3 = -100$.)

We assume the “flow conservation” constraints: the total number of cars per hour flowing into each intersection is zero. For example at intersection $B$, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$
\begin{align*}
  t_1 + t_3 &= 0 \\
  t_2 - t_1 &= 0 \\
  -t_3 - t_2 &= 0
\end{align*}
$$

As mentioned earlier, we can place sensors on a road to measure the flow through it. But, we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.
Suppose for the network above we have one sensor reading, \( t_1 = 10 \). Can we figure out the flows along the other roads? (That is, the values of \( t_2 \) and \( t_3 \)).

**Solution:** Yes, since we know that \( t_1 = t_2 = -t_3 \), so we must have \( t_2 = 10 \) and \( t_3 = -10 \).

(b) Now suppose we have a larger network, as shown in Figure 4.

![Figure 4: A larger road network.](image)

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads AD and BA. A Stanford student claims that we need two sensors placed on the roads CB and BA. Is it possible to determine all traffic flows with the Berkeley student’s suggestion? How about the Stanford student’s suggestion?

**Solution:** The Stanford student is wrong (obviously). Observing \( t_1 \) and \( t_2 \) is not sufficient, as \( t_3 , t_4 \) and \( t_5 \) can still not be uniquely determined. Specifically, for any \( t \in \mathbb{R} \), the following flow satisfies the constraints and the measurements:

\[
\begin{align*}
    t_4 &= t \\
    t_5 &= t \\
    t_3 &= t - t_1
\end{align*}
\]

On the other hand, if we’re given \( t_1 \) and \( t_4 \), we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. We know that \( t_2 \) is the same as \( t_1 \) and \( t_4 \) is the same as \( t_5 \), since the flow going into B and D must equal to the flow going out. The flow into A, \( t_1 + t_3 \), must equal the flow going out, \( t_4 \), so:

\[
\begin{align*}
    t_2 &= t_1 \\
    t_5 &= t_4 \\
    t_3 &= t_4 - t_1
\end{align*}
\]

This is related to the fact that \( t_1 \) and \( t_4 \) are parts of different cycles in the graph, whereas \( t_1 \) and \( t_2 \) are in the same cycle, so measuring both of them would not give additional information.

(c) Suppose we write the traffic flow on all roads as a vector \( \vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} \). Show that the set of valid flows (which satisfy the conservation constraints) form a subspace. Then, determine the subspace of traffic flows for the network of Figure 4. Specifically, express this space as the span of two linearly independent vectors.

**Solution:**

(Hint: Use the claim of the correct student in the previous part.)
Suppose we have a set of valid flows $\vec{t}$. Then for any intersection, the net flow into it is the same as the net flow out of it. If we scale $\vec{t}$ by a constant $a$, each $t_i$ will also get scaled by $a$. The net flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{t}_1$ and $\vec{t}_2$ to get $\vec{t} = \vec{t}_1 + \vec{t}_2$. For any intersection $I$,

$$\text{net flow into } I = \text{net flow into } I \text{ from } \vec{t}_1 + \text{net flow into } I \text{ from } \vec{t}_2$$

$$\text{net flow out of } I = \text{net flow out of } I \text{ from } \vec{t}_1 + \text{net flow out of } I \text{ from } \vec{t}_2$$

Since the net flow into $I$ from $\vec{t}_1$ is the same as net flow out of $I$ from $\vec{t}_1$ and similarly for $\vec{t}_2$, the net flow into $I$ is the same as the net flow out of $I$. Therefore the sum of any two valid flows is still a valid flow. Therefore the set of valid flows forms a subspace.

To determine the subspace of traffic flows for the above network, use the solution in the previous part to see what $\vec{t}$ looks like in terms of $t_1 = \alpha$ and $t_4 = \beta$:

$$\vec{t} = \begin{bmatrix} \alpha \\ \alpha \\ \beta - \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \alpha \vec{u}_1 + \beta \vec{u}_2$$

Clearly, $\vec{u}_1$ and $\vec{u}_2$ are linearly independent, and the space of all possible traffic flows is the span of them.

(d) We would like a more general way of determining the possible traffic flows in a network. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation. Find a $(4 \times 5)$ matrix $B$ such that the equation $B\vec{t} = \vec{0}$:

$$\begin{bmatrix} B \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network of Figure 4.

(Hint: Each row is the constraint of an intersection. You can construct $B$ using only 0, 1, $-1$ entries.) This matrix’s transpose is called the incidence matrix. What does each row of this matrix represent? What does each column of this matrix represent?

Solution:

$$B = \begin{bmatrix} -1 & 0 & -1 & +1 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix}$$

(A, B, C, D, t_1, t_2, t_3, t_4, t_5)

(The rows of this matrix can be in any order.) Each row represents an intersection, and each column represents a road between two intersections. Each $-1$ on a row represents a road flowing into an intersection, and each $+1$ represents a road flowing out of an intersection. Each $+1$ in a column represents the source intersection of a road, and each $-1$ in a column represents the destination intersection of a road.)
(e) Notice that the set of all vectors $\vec{t}$ which satisfy $B\vec{t} = \vec{0}$ is exactly the null space of the matrix $B$. That is, we can find all valid traffic flows by computing the null space of $B$. Use Gaussian Elimination to determine the dimension of the null space of $B$, and compute a basis for the null space. (You may use a computer to compute the reduced row echelon form.) Does this match your answer to part (c)? Can you interpret the dimension of the null space of $B$, for the road networks of Figure 3 and Figure 4?

**Solution:** After row-reducing, we get the following matrix:

$$
\begin{bmatrix}
+1 & 0 & +1 & 0 & -1 \\
0 & +1 & +1 & 0 & -1 \\
0 & 0 & 0 & +1 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Since the column rank is 3, the dimension of the null space is 2. We can find the following basis (using that observation that we had made that connects the basis vectors to the row-reduced matrix. Here, $t_3$ and $t_5$ are the free variables and so get 1s with the rest coming from sign flipping.) for the null-space:

$$
[ -1 \\
-1 \\
1 \\
0 \\
0 ] + [ 1 \\
1 \\
0 \\
1 \\
1 ]
$$

This does not match the answer in the earlier part because these are two different bases. But the null space they span is the same.

By itself, the first vector weighted by $a$ clearly is a vector corresponding to a small cycle in the graph. But the second one $b$ is going around a bigger cycle. These two cycles are still independent of each other though. This is why the dimension of the null space can be interpreted as the number of “independent cycles” in the graph.

It is fine to give yourself full credit as long as you found a basis for the null space. It doesn’t have to be this particular one.

(f) (PRACTICE) Now let us analyze general road networks. Say there is a road network graph $G$, with incidence matrix $B_G$. If $B_G^T$ has a $k$-dimensional null space, does this mean measuring the flows along any $k$ roads is always sufficient to recover the exact flows? Prove or give a counterexample.

**Solution:** No, consider the network of Figure 4. The corresponding incidence matrix’s transpose has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring $t_1$ and $t_2$ (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(g) (PRACTICE) Let $G$ be a network of $n$ roads, with incidence matrix $B_G$ whose transpose has a $k$-dimensional null space. We would like to characterize exactly when measuring the flows along a set of $k$ roads is sufficient to recover the exact flow along all roads. To do this, it will help to generalize the problem, and consider measuring linear combinations of flows. If $\vec{t}$ is a traffic flow vector, assume we can measure linear combinations $\vec{m}_i^T \vec{t}$ for some vectors $\vec{m}_i$. Then making $k$ measurements is equivalent to observing the vector $\vec{M} \vec{t}$ for some $(k \times n)$ “measurement matrix” $\vec{M}$ (consisting of rows $\vec{m}_i^T$).

For example, for the network of Figure 4 the measurement matrix corresponding to measuring $t_1$ and $t_4$ (as the Berkeley student suggests) is:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$
Similarly, the measurement matrix corresponding to measuring $t_1$ and $t_2$ (as the Stanford student suggests) is:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For general networks $G$ and measurements $M$, give a condition for when the exact traffic flows can be recovered, in terms of the null space of $M$ and the null space of $B_G^T$. 

(Hint: Recovery will fail iff there are two valid flows with the same measurements. Can you express this in terms of the null spaces of $M$ and $B_G^T$?)

**Solution:** As stated in the hint, we cannot uniquely determine the flow iff there are two valid flows that yield the same set of measurements. That is, there should not be two distinct valid flows $\bar{t}_1$ and $\bar{t}_2$ such that $M\bar{t}_1 = M\bar{t}_2$. Or equivalently, such that $M(\bar{t}_1 - \bar{t}_2) = 0$.

The set of valid flows is the null space of $B_G^T$, denoted $\text{Null}(B_G^T)$. So recovery fails if $M(\bar{t}_1 - \bar{t}_2) = \vec{0}$ for some $\bar{t}_1, \bar{t}_2 \in \text{Null}(B_G^T)$, with $\bar{t}_1 \neq \bar{t}_2$. The set of valid flows is a subspace, so we can equivalently state this as: Recovery fails iff $M\bar{t} = \vec{0}$ for some $\bar{t} \neq \vec{0}, \bar{t} \in \text{Null}(B_G^T)$.

In other words, there should be no vector $\bar{t} \neq \vec{0}$ that is both in the null space of $B_G^T$ and in the null space of $M$.

This can also be stated as: We can recover the exact traffic flows iff the null space of $B_G^T$ does not non-trivially intersect the null space of $M$.

Full credit for stating any condition that is equivalent to this, using the null spaces of $M$ and $B_G^T$.

(h) (PRACTICE) Express the condition of the previous part in a way that can be checked computationally. For example, suppose we are given a huge road network $G$ of all roads in Berkeley, and we want to find if our measurements $M$ are sufficient to recover the flows.

(Hint: Consider a matrix $U$ whose columns form a basis of the null space of $B_G^T$. Then $\{U\vec{x} : \vec{x} \in \mathbb{R}^k\}$ is exactly the set of all possible traffic flows. How can we represent measurements on these flows?)

**Solution:** Let $U$ be a matrix whose columns form a basis of the null space of $B_G^T$. Then, as in the hint, the set $\{U\vec{x} : \vec{x} \in \mathbb{R}^k\}$ is exactly the set of all possible traffic flows.

For a given valid flow $\bar{t} = U\vec{x}$, the result of measuring this flow is $M\bar{t} = MU\vec{x}$. Now, recovering the exact flow from our measurements is equivalent to recovering $\vec{x}$ from $MU\vec{x}$. Notice that the matrix $MU$ is $(k \times k)$, so to we can recover the exact flows iff $MU$ is invertible. This condition can be easily checked computationally (for example, by row-reducing $MU$).

**Remark:** Notice how defining the matrix $U$ allowed us to work with flows in terms of their low-dimensional representations ($\vec{x}$), instead of dealing directly with all their components.

(i) (PRACTICE) If the incidence matrix’s transpose $B_G^T$ has a $k$-dimensional null space, does this mean we can always pick a set of $k$ roads such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

**Solution:** Yes.

Let $U$ be a matrix whose columns form a basis of the null space of $B_G^T$, as above. The $k$ columns of $U$ are linearly independent, since they form a basis. Since there are $k$ linearly independent columns, when we run Gaussian elimination on $U$, we must get $k$ pivots. (Recall that “pivot” is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore the row-space of $U$ is $k$ dimensional, since there are some $k$ linearly independent rows in $U$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because: For a given valid flow $\bar{t} = U\vec{x}$, the results of measuring this flow vector is $U^{(k)}\vec{x}$, where the matrix $U^{(k)}$ is some $k$ linearly independent rows of $U$. By construction, the $(k \times k)$
3. Cell Phone Battery

As great as smartphones are, one of the main gripes about them is that they need to be recharged too often. Suppose a Samsung Galaxy S3 requires about 0.4 W to maintain a signal as well as its regular activities (dominated by the display and backlight in many cases). The battery provides 2200 mAh at a voltage of 3.8V until it is completely discharged.

(a) How long will one full charge last you?

Solution: 400 mW of power at 3.8V is about 105.26mA of current. A battery that can provide 1mAh can provide 1mA for an hour, so our 2200mAh battery can source 105.26mA for 2200/105.26 = 20.9 hours, almost a full day.

An alternative approach is to say 2200 mAh at 3.8V is 2200 × 3.8 = 8360 milliamp-hours. 0.4 W is 400 mW, so 8360/400 = 20.9 hours is how long the charge will last.

(b) Suppose the cell phone battery is completely discharged and you want to recharge it completely. How much energy (in J) is this? How much charge (in C) must be pumped through the battery?

Solution: The battery is rated for 2200 mAh at 3.8V, which gives 2200 × 3.8 = 8360 milliamp-hours. A joule is equivalent to watt-second, and there are 3600 seconds in an hour, so our battery has 8360 × 3600 = 30,096,000 mJ, or 30,096 J. A milliamp-hour is 0.001 coulombs per second, so a milliamp-hour is 0.001 × 3600 = 3.6 coulombs. Then a 2200 mAh battery needs 3.6 × 2200 = 7,920 coulombs of charge to be moved to be fully recharged. An electron has charge approximately 1.602 × 10^{-19} C, so 7,920 coulombs is 7920/1.602 × 10^{-19} = 4.94 × 10^{22} electrons.

(c) Suppose PG&E charges $0.16 per kWh. Every day, you completely discharge the battery and recharge it at night. How much will recharging cost you for the month of March (31 days)?

Solution: 2200 mAh at 3.8V is 2200 × 3.8 = 8,360 milliamp-hours, or 0.00836 kWh. At $0.16 per kWh, that is 0.16 × 0.00836 dollars per day, or 0.16 × 0.00836 × 31 = 0.0414656, or about 4 cents a month.

(d) You are fed up with PG&E, gas companies, and Duracell/Energizer/etc. You want to generate your own energy and decide to buy a small solar cell for $1.50. It delivers 40 mA at 0.5 V in bright sunlight. Unfortunately, now you can only charge your phone when the sun is up. Using one solar cell, do you think there is enough time to charge a completely discharged phone every day? How many cells would you need to charge a completely discharged battery in an hour? How much will it cost you per joule if you have one solar cell that works for 10 years (assuming you can charge for 16 hours a day)? Do you think this is a good option?

Solution: One solar cell provides 40 mA at 0.5 V in bright sunlight, which is 40 × 0.5 = 20 mW. Note that this is considerably less than the phone’s power consumption of 400 mW, so we should expect the solar panel to take much longer to charge than the ≈ 1 day of battery life. In fact, 2200 × 3.8 × (20)^{-1} = 418 hours to charge using one panel, so even if the sun was bright all day, it would not be enough to charge the phone. It would take 418 panels to charge the battery in an hour. The total amount of energy collected over 10 years, assuming 16 hours of operation per day, is approximately (neglecting leap-years, etc.) 10 × 365 × 16 × 40 × 0.5 = 1,168,000 mWh, or 1,168 kWh. That is 1.50/1.168 ≈ 1.28 dollars per kWh, or 1.50 × (1.168 × 3600 × 1000) = 3.57 × 10^{-7} dollars per joule. This particular solar panel doesn’t provide enough power to charge the phone’s battery during the daylight hours (with a...
reasonable number of panels). However, if the panel lasts long enough, the cost per joule is actually pretty reasonable.

(e) The battery has a lot of internal circuitry that prevents it from getting overcharged (and possibly exploding!) as well as transferring power into the chemical reactions used to store energy. We will model this internal circuitry as being one resistor with resistance $R_{\text{bat}}$, which you can set to any non-negative value you want. Furthermore, we’ll assume that all the energy dissipated across $R_{\text{bat}}$ goes to recharging the battery. Suppose the wall plug and wire can be modeled as a 5V voltage source and 200 mΩ resistor, as pictured in Fig. 5. What is the power dissipated across $R_{\text{bat}}$ for $R_{\text{bat}} = 1\text{m} \Omega$, 1 Ω, and 10 kΩ? How long will the battery take to charge for each of those values of $R_{\text{bat}}$?

![Figure 5: Model of wall plug, wire, and battery.](image)

**Solution:** The energy stored in the battery is 2200mAh at 3.8V, which is $2.2 \times 3.8 = 8.36\text{Wh}$. We can find time to charge by dividing this energy by power in W to get time in hours.

For $R_{\text{bat}} = 1\text{m} \Omega$, the total resistance seen by the battery is $1\text{m} \Omega + 200\text{m} \Omega = 201\text{m} \Omega$ (because the wire and $R_{\text{bat}}$ are in series), so by Ohm’s law, the current is $5/0.201 = 24.88\text{A}$. The voltage drop across $R_{\text{bat}}$ is (again by Ohm’s law) $24.88 \times 0.001 = 0.025\text{V}$. Then power is $0.025 \times 24.88 = 0.622\text{W}$ and the total time to charge the battery is $8.36/0.622 = 13.44\text{hours}$.

Similarly, for 1 Ω, the total resistance seen by the battery is $1 + 0.200 = 1.2\Omega$, the current through the battery is $5/1.2 = 4.17\text{A}$, and the voltage across the battery is by Ohm’s law $4.17 \times 1 = 4.17\text{V}$. Then the power is $4.17 \times 4.17 = 17.39\text{W}$ and the total time to charge the battery is $8.36/17.39 = 0.48\text{hours}$, about 29 minutes.

For 10 kΩ, the total resistance seen by the battery is $10000 + 0.200 = 10000.2\Omega$, the current through the battery is $5/10000.2 \approx 0.5\text{mA}$, and the voltage across the battery is by Ohm’s law $0.5\text{mA} \times 10\text{k} \Omega \approx 5\text{V}$ (up to 2 significant figures). Then the power is $5\text{V} \times 0.5\text{mA} = 2.5\text{mW}$ and the total time to charge the battery is $8.36/0.0025 = 3344\text{hours}$.

4. **Mechanical Circuits**

Find the voltages across and currents flowing through all the resistors.
Solution: Approach 1 – KCL / KVL:

First, label all the ‘junctions’ or ‘nodes’:

Now set up your KCL / KVL equations:

\[ i_1 = i_2 + i_{34} \]
\[ v_1 = i_1 \cdot R_1 \]
\[ v_2 = i_2 \cdot R_2 \]
\[ v_3 = i_{34} \cdot R_3 \]
\[ v_4 = i_{34} \cdot R_4 \]
\[ v_2 = v_3 + v_4 = i_{34} \cdot (R_3 + R_4) \]
\[ 10 = v_1 + v_2 = v_1 + v_3 + v_4 \]

You can solve as a system of equations:

\[ \frac{10 - v_2}{R_1} = \frac{v_2}{R_2} + \frac{v_2}{R_3 + R_4} \]
\[
\frac{10}{R_1} = v_2 \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3 + R_4} \right) \\
\frac{10}{3} = v_2 \left( \frac{1}{3} + \frac{1}{3} + \frac{1}{4.5 + 1.5} \right) = v_2 \cdot \frac{5}{6} \implies v_2 = 4V \\
v_1 = 10 - v_2 = 6V \\
i_{34} = \frac{4}{4.5 + 1.5} = \frac{2}{3} \text{A} \\
v_3 = i_{34} R_3 = 3V \\
v_4 = i_{34} R_4 = 1V
\]

Alternatively, you could set it up as a matrix and use IPython / NumPy to solve.

\[
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -4.5 & 0 & 0 & 1 & 0 \\
0 & 0 & -1.5 & 0 & 0 & 0 & 1 \\
0 & 0 & -6 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
i_1 \\
i_2 \\
i_{34} \\
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
10
\end{bmatrix}
\]

This returns the array:

\[
\begin{bmatrix}
i_1 \\
i_2 \\
i_{34} \\
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
\frac{4}{3} \\
\frac{2}{3} \\
6 \\
4 \\
3 \\
1
\end{bmatrix}
\]

**Approach 2:**

We will first calculate the effective resistance seen from the voltage source to find the current supplied by the voltage source. The resistances \(R_3\) and \(R_4\) are in series hence have effective resistance of 6\(\Omega\). They are connected in parallel to a \(R_2\) resistance yielding an effective resistance of \(\left( \frac{1}{6} + \frac{1}{3} \right)^{-1} = 2\Omega\).

This resulting effective resistance is in series to \(R_1\), yielding an effective resistance of 5\(\Omega\). Hence the current supplied by the voltage source is

\[
\frac{10V}{5\Omega} = 2A.
\]

Let us denote the voltage drop across \(R_i\) as \(V_i\), and the current flowing through \(R_i\) and \(I_i\). We have \(I_1 = 2A\) current flowing through \(R_1\), hence we have \(V_1 = 6V\). The remaining voltage \(10V - 6V = 4V\) is across both \(R_2\) and the sequence of resistors \(R_3\) and \(R_4\). Hence, \(I_2 = \frac{4V}{3\Omega} = \frac{4}{3}A\). Furthermore, \(I_3 = \frac{4V}{6\Omega} = \frac{2}{3}A\). Combining
all, we get the following voltages and currents:

\[ I_1 = 2A, \]
\[ I_2 = \frac{4}{3}A, \]
\[ I_3 = I_4 = 2\frac{2}{3}A, \]
\[ V_1 = 6V, \]
\[ V_2 = 4V, \]
\[ V_3 = 3V, \]
\[ V_4 = 1V. \]

5. Midterm Problem 3
Redo Midterm Problem 3.

**Solution:** See midterm solutions.

6. Midterm Problem 4
Redo Midterm Problem 4.

**Solution:** See midterm solutions.

7. Midterm Problem 5
Redo Midterm Problem 5.

**Solution:** See midterm solutions.

8. Midterm Problem 6
Redo Midterm Problem 6.

**Solution:** See midterm solutions.

9. Midterm Problem 7
Redo Midterm Problem 7.

**Solution:** See midterm solutions.

10. Midterm Problem 8
Redo Midterm Problem 8.

**Solution:** See midterm solutions.

11. Midterm Problem 9
Redo Midterm Problem 9.

**Solution:** See midterm solutions.

12. Homework process and study group

Who else did you work with on this homework? List names and student ID’s. (In case of hw party, you can also just describe the group.) How did you work on this homework?

Working in groups of 3-5 will earn credit for your participation grade.

**Solution:**
I worked on this homework with...
I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...
Then I went to homework party for a few hours, where I finished the homework.