Fun with Stacked Caps

Consider a capacitor circuit with switches. Suppose that in Phase 1, the circuit looks like the circuit in Figure 1:

\[ V_s \quad \begin{array}{c} Q_{A1} \\ \hline \end{array} C_A \quad \begin{array}{c} Q_{B1} \\ \hline \end{array} C_B \]

Figure 1: Phase 1

Since the voltage source forces \( V_s \) across each capacitor, we do not need the initial state of the capacitors to determine the charges.

\[
Q_{A1} = C_A V_s \\
Q_{B1} = C_B V_s
\]

Suppose we flip switches such that the circuit looks like Figure 2:

\[ C_A \quad \begin{array}{c} Q_{A2} \\ \hline \end{array} V_{A2} \]
\[ \downarrow D \]
\[ \begin{array}{c} Q_{B2} \\ \hline \end{array} V_{B2} \]

\[ C_B \]

Figure 2: Phase 2

We note that there are no paths for current to flow or charge to move.
Therefore:

\[ Q_{A2} = Q_{A1}, \quad V_{A2} = \frac{Q_{A1}}{C_A} = V_s \]
\[ Q_{B2} = Q_{B1}, \quad V_{B2} = \frac{Q_{B1}}{C_A} = V_s \]

Suppose we now attach a voltage source \( V_x \) to this circuit:

\[ \Delta Q \]
\[ C_A \]
\[ Q_{A3} \]
\[ V_{A3} \]
\[ V_x \]
\[ C_B \]
\[ Q_{B3} \]
\[ V_{B3} \]

**Figure 3: Phase 3**

\( \Delta Q \) denotes the change of charge on \( C_A \). Therefore:

\[ \Delta Q = Q_{A3} - Q_{A2} \]

From charge conservation on node D, we obtain:

\[-Q_{A3} + Q_{B3} = -Q_{A2} + Q_{B2} \]

By rearranging the above two equations, we obtain:

\[ \Delta Q = Q_{B3} - Q_{B2} \]

We conclude that a change in charge is the same on both capacitors. This change in charge \( \Delta Q \) is the charge supplied by the new voltage source \( V_x \).

Using KVL:

\[ V_x = V_{A3} + V_{B3} \]
\[ V_x = \frac{Q_{A3}}{C_A} + \frac{Q_{B3}}{C_B} \]
\[ V_x = \frac{Q_{A2} + \Delta Q}{C_A} + \frac{Q_{B2} + \Delta Q}{C_B} \]

\[ V_x = \frac{Q_{A1}}{C_A} + \frac{Q_{B1}}{C_B} + \Delta Q \left( \frac{1}{C_A} + \frac{1}{C_B} \right) \]
Note that only if \( Q_{A1} = Q_{B1} = 0 \implies \Delta Q = V_x \cdot (C_A \parallel C_B) \)

Otherwise, we need to take into account the prior charge on caps \((Q_{A1}, Q_{B1})\) as in (1). In general:

\[
\Delta Q = \left( V_x - \frac{Q_{A1}}{C_A} - \frac{Q_{B1}}{C_B} \right) \cdot (C_A \parallel C_B)
\]

For this example:

\[
\Delta Q = (V_x - V_s - V_s) \cdot (C_A \parallel C_B) \\
\Delta Q = (V_x - 2V_s) (C_A \parallel C_B)
\]