1. Mechanical Gram-Schmidt (Fall 2016 Final)

(a) Use Gram-Schmidt to find an orthonormal basis for the following three vectors.

\[ \vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix} \]

**Answer:**

A valid basis \( \mathcal{B} \) is given by:

\[ \mathcal{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\} \]

(b) Express \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) as vectors in the basis you found in part (a).

**Answer:**

Using the basis above:

\[ [\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\vec{v}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \end{bmatrix} \]

2. Gram-Schmidt Properties

(a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the following set of vectors.

\[ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

Perform Gram-Schmidt on these vectors first in the order \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) and then in the order \( \vec{v}_3, \vec{v}_2, \vec{v}_1 \). Do you get the same answer?

**Answer:**

If we start with \( \vec{v}_1 \), we get the basis vectors:

\[ \vec{e}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ \vec{e}_2 = \vec{v}_2 - \frac{(\vec{e}_1, \vec{v}_2)}{\|\vec{e}_1\|^2} \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]
3. Orthonormal Projections

(a) Suppose that the $n \times m$ matrix $A$ has linearly independent columns. The vector $\vec{y}$ in $\mathbb{R}^n$ is not in the subspace spanned by the columns of $A$. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $A$ is $A(A^TA)^{-1}A^T\vec{y}$.

Answer:
When finding a projection onto a subspace, we’re trying to find the “closest” vector in that subspace. This can be found by first finding $\vec{x}$ that minimizes $||\vec{y} - A\vec{x}||$. From least squares, we know that $\vec{x} = (A^TA)^{-1}A^T\vec{y}$. The projection of $\vec{y}$ onto the columns of $A$ is then $A\vec{x} = A(A^TA)^{-1}A^T\vec{y}$.

(b) Now suppose that we perform Gram-Schmidt on $A$ to get a new matrix $Q$. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $Q$ is now $QQ^T\vec{y}$.

Answer:
Plugging in $Q$ for $A$ in the above expression, we get

$$Q(Q^TQ)^{-1}Q^T\vec{y} = QQ^T\vec{y}$$

Since we performed Gram-Schmidt on the columns of $A$ to get $Q$, we know that $Q$ is a matrix with orthonormal columns, so $Q^TQ = I$. 

3. Orthonormal Projections

(a) Now suppose that we perform Gram-Schmidt on $\vec{y}$ onto the columns of $A$.

(b) What happens when we perform Gram-Schmidt on a set of $n$ vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$, where only $n - 1$ of them are linearly independent?

Answer:
This can be explained geometrically. Remember that Gram-Schmidt is basically about finding the “error” vector between a vector and its projection onto a subspace. If the vector is already in the subspace, then the “error” vector is $\vec{0}$.