Module 3  Lec 1

- positioning/locationing
- norms
- inner products

Positioning with GPS: (global positioning system)

How do dist. help positioning?
How to meas. distances?
How many satellites do I need?

Let's simplify to 1D:

Assume I can meas. d_1, does this uniquely give position?
No, cause could be d_1 to
direction on either side...

What else do I need? direction (vector!)

OR? A second known landmark!

What are requirements on:
Landmark L1, L2?
- known
- d_1 ≠ d_2

How do things change in 2D?

Could I be anywhere on this circle?
⇒ More ambiguity in 2-D.

Now what to do?
⇒ add a 3rd Landmark!

Now correct position is only:
solution!

Could I do it in only two? Not in general

Need: 3 known landmarks ⇒ "triangulation"
- dist. from each
- not collinear (prove later) "triangulation"
Problem: How can I get the distance measurement?
- Ruler? Not practical
- Send a robot at speed $v$, see how long it takes ($d = vt$)?
  Robots are not practical
- Send a signal wave: EM waves (light, microwave) [**GPS:** $3 \times 10^8$ m/s]
  Sound waves [**APS:** $340$ m/s]  
  \[ \text{Assume for now we have clocks @ Landmarks and receiver (me)} \]
  \[ \text{that are synced.} \]

\[ \text{GPS/APS: the beacons "sing songs" in a periodic loop, transmitting in all directions equally.} \]
\[ \text{One-way} \]
\[ \text{Rx listens to the songs (from many beacons may be)} \]

Example:

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>8</td>
</tr>
</tbody>
</table>

"Song" is 0, 1, 2, 3, 0, 1, 2, 3, 0,...

time = samples

Assume discrete time.

So, if signal takes 2 samples of time to get from beacon to receiver (Rx),
then it will be delayed by 2 samples.

What if delay = 4 samples?
Cannot distinguish from 0, 8, 12, 16...
To avoid this, use **ENGINEERING**

How?
Make sure max distance is song length.
I.e.: if 4 samples = 4 seconds
and $v = 200$ m/s, then max dist.

\[ w \text{ no ambiguity is } d = vt \]
\[ =\left(100\text{ m/s}\right)\left(4\text{ s}\right) = 400\text{ m} \]

**GPS:** 1 ms $\rightarrow$ 300 km
**APS:** 23 ms $\rightarrow$ 55 m
write beacon song as a vector:
\[
\tilde{S} = \begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
s(0) \\
s(1) \\
s(2) \\
s(3)
\end{bmatrix}
\]
song is N-periodic (N samples is period of song). Here N = 4

Received signal is delayed:
\[
\tilde{r} = \begin{bmatrix}
2 \\
3 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
r(0) \\
r(1) \\
r(2) \\
r(3)
\end{bmatrix}
\]

define shifted vectors:
\[
\tilde{S}_0 = \begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix} \quad \tilde{S}_1 = \begin{bmatrix}
3 \\
0 \\
1 \\
2
\end{bmatrix} \quad \tilde{S}_2 = \begin{bmatrix}
2 \\
3 \\
0 \\
1
\end{bmatrix} \quad \tilde{S}_3 = \begin{bmatrix}
1 \\
2 \\
3 \\
0
\end{bmatrix}
\]

this is what we receive after 2 sample delay.

How many possible shifts?
\[
\{0, 1, 2 \ldots N-1\}
\]

How should I find the shift, \(k = 2\) from the received signal?

- Just look at \(r(0)\) and see which entry of \(\tilde{S}\) it is? not very robust
- Compare against dictionary of shifted vectors \(\{\tilde{S}_0, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3\}\)? may not be equal if errors/noise
- What we will do is
\[
\begin{align*}
||\tilde{r} - \tilde{S}_0|| \\
||\tilde{r} - \tilde{S}_1|| \\
||\tilde{r} - \tilde{S}_2|| \\
||\tilde{r} - \tilde{S}_3||
\end{align*}
\]
figure out which one is smallest value \(\rightarrow\) delay \(k\)
\[
||\tilde{r} - \tilde{S}_k|| = 0
\]

What is this notation? norm

(Euclidian) norm of a vector \(\tilde{X}\) is
\[
||\tilde{X}||_2 = ||\tilde{X}|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}
\]

Intuitively, what is norm?
- magnitude (length) of vector - distance in \(\mathbb{R}^2, \mathbb{R}^3\)

So we're looking for min. dist. between \(\tilde{r}\) and \(\tilde{S}_k\) \(\rightarrow\) ERROR

\[
||\tilde{r} - \tilde{S}_k||_2 = \sqrt{2^2 + 1^2} = \sqrt{5}
\]
Norm has square roots which are annoying, so just use error $^2$

Why can I do that? doesn't change which is min

$$\| \tilde{e}_i \|^2 = \| \tilde{r} - \tilde{s}_i \|^2 = (\tilde{r} - \tilde{s}_i)^T (\tilde{r} - \tilde{s}_i)$$

$$= (\tilde{r} - \tilde{s}_i)^T (\tilde{r} - \tilde{s}_i)$$

$$= \tilde{r}^T \tilde{r} - (\tilde{r}^T \tilde{s}_i) - (\tilde{s}_i^T \tilde{r}) + \tilde{s}_i^T \tilde{s}_i$$

$$= \| \tilde{r} \|^2 + \| \tilde{s}_i \|^2 - (\tilde{s}_i^T \tilde{r} + \tilde{r}^T \tilde{s}_i)$$

Which terms change in $j$?

No, cause shifted version of something $\rightarrow$ same length

Check: $\| \tilde{s}_1 \|^2 = S(0)^2 + S(1)^2 + ... S(n)^2$

Ex. $\| \tilde{s}_1 \|^2 = S(-1)^2 + S(0)^2 + ... S(n-2)^2 = \| \tilde{s} \|^2$

Want to minimize $\| \tilde{e}_i \|^2 = 2\| \tilde{s} \|^2 - \left( \tilde{s}_i^T \tilde{r} - \tilde{r}_i^T \tilde{s}_i \right)$

Equiv. to maximize this

What is this?

If $\tilde{a} = \tilde{s}_i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\tilde{b} = \tilde{r} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then $\tilde{a}^T \tilde{b} = (a_1 b_1 + a_2 b_2 + ... + a_n b_n)$

Element-wise multiplication

Euclidian "Inner Product"

$\tilde{a}^T \tilde{b} = \langle \tilde{a}, \tilde{b} \rangle$ aka "dot product"

$\tilde{a} \cdot \tilde{b}$

In 2D:

$\tilde{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$\| \tilde{a} \|^2 = a_1^2 + a_2^2$  \theta_a angle

$\| \tilde{b} \|^2 = b_1^2 + b_2^2$  \theta_b angle

Can also write:

$\tilde{a} = \begin{bmatrix} \| \tilde{a} \| \cos \theta_a \\ \| \tilde{a} \| \sin \theta_a \end{bmatrix}$

$\tilde{b} = \begin{bmatrix} \| \tilde{b} \| \cos \theta_b \\ \| \tilde{b} \| \sin \theta_b \end{bmatrix}$

Let's compute inner product:

$\langle \tilde{a}, \tilde{b} \rangle = a_1 b_1 + a_2 b_2$

$= \| \tilde{a} \| \| \tilde{b} \| \cos \theta_a \| \tilde{b} \| \cos \theta_b + \| \| \tilde{a} \| \| \tilde{b} \| \sin \theta_a \| \tilde{b} \| \sin \theta_b$

$\theta_a - \theta_b$

So measures how "aligned/similar" vectors are when colinear $\rightarrow$ large when $\tilde{a} \rightarrow \tilde{b}$
some rules about inner products:

1. \( \langle x, y \rangle = \langle y, x \rangle \)
2. \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \)
3. \( \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \)
4. \( \langle x, x \rangle \geq 0 \)
5. \( \langle x, x \rangle = 0 \) iff \( x = 0 \)

What does \( \langle a, b \rangle = -1 \) mean? anti-similar (opp. dir.)

What is max. of inner product?
\( \langle \hat{a}, b \rangle \leq \|a\| \cdot \|b\| \)

Cauchy-Schwarz Inequality

Projections "project \( \hat{b} \) onto subspace spanned by \( \hat{a} \)"

Scary! \( \rightarrow \) go to geometry

- Projected vector is component of \( \hat{b} \) along direction of \( \hat{a} \)
- What does it have to do with inner product?
- Looking for collinear component \( \rightarrow \) largest inner prod.
- Or look for error \( \rightarrow \) inner prod. = 0

Perp. vectors have inner prod. = 0:
\( \langle \hat{e}, \hat{a} \rangle = 0 \)
\( \langle \hat{b} - \hat{e}, \hat{a} \rangle = 0 \)
\( \langle \hat{b}, \hat{a} \rangle - \langle \hat{e}, \hat{a} \rangle = 0 \)
\( \langle \hat{b}, \hat{a} \rangle = \langle \alpha \hat{a}, \hat{a} \rangle \)
\( \langle \hat{b}, \hat{a} \rangle = \alpha \langle \hat{a}, \hat{a} \rangle \)
\( \alpha = \frac{\langle \hat{b}, \hat{a} \rangle}{\| \hat{a} \|^2} \)

So then \( \hat{b} = \frac{\langle \hat{b}, \hat{a} \rangle}{\| \hat{a} \|^2} \hat{a} \)

Scaling factor