1. Timer Circuit

In this problem, we will walk through the timer circuit, shown below, similar to the one seen in lecture. The circuit is shown below. All resistors have a resistance of 1 kΩ and \( C_1 = 1 \mu\text{F} \).

![Circuit Diagram]

(a) Find the current through the capacitor \( C_1 \) in terms of the voltage \( V_3 \) and the resistor \( R_1 \).

**Answer:**

For an op-amp, no current flows into the input terminals. Therefore, all the current through \( R_1 \) must flow through \( C_1 \). Applying the Golden Rules, we know that \( v_+ = v_- = 0 \text{V} \).

\[ i_{R_1} = i_{C_1} = \frac{v_3}{R_1} \]

(b) Suppose that at time \( t = 0 \), \( C_1 \) is uncharged. Find the voltage \( v_1 \) in terms of \( t \), \( v_3 \), and \( R_1 \). What is the maximum \(|v_1|\) could be?

**Answer:**

Recall the voltage across a capacitor is related to the charge on the capacitor, that is \( Q = CV \). Current is related to charge with the equation \( I = \frac{dQ}{dt} \).

\[ v_{C_1} = \frac{Q}{C_1} = \frac{1}{C_1} \int I \, dt = \frac{v_3}{R_1 C_1} t = \frac{v_3}{1 \text{ms}} t \]

Note that a \( \Omega \text{F} \) is a second. Because the current is flowing into the capacitor, as the voltage across the capacitor increases, the output voltage decreases.

\[ v_1 = -v_{C_1} = -\frac{v_3}{1 \text{ms}} t \]
The maximum or minimum for $v_1$ is the top or bottom supply rail, so either $+1$ V or $-1$ V. Therefore, the maximum $|v_1| = 1$ V.

(c) How is $v_2$ related to $v_1$? What is the voltage $v_2$?

**Answer:**

$O_2$ is an inverting amplifier. The output voltage $v_2$ is equal to $-v_1$.

$$v_2 = \frac{v_3}{1 \text{ms}} t$$

$O_3$ is not connected in negative feedback. However, we can analyze its behavior by considering it to be a comparator. Let’s independently analyze the circuit in the two possible outputs of the comparator, when $v_3 = 1$ V and when $v_3 = -1$ V.

(d) Assume that the output of the comparator $v_3$ has railed to the top rail. With this value of $v_3$, what is $v_2$ as a function of time? What is the voltage at the positive input of $O_3$? At what time will the two inputs of the comparator be equal?

**Answer:**

With $v_3$ at the top rail, $v_2$ is $\frac{t}{1 \text{ms}}$ V. The voltage at the positive input of the opamp is 0.5 V because of $R_5$ and $R_4$. Therefore, when $t = 0.5$ ms, $v_2 = 0.5$ V.

(e) Now assume that the reverse occurs, that is, the output of the comparator has railed to the bottom rail. Repeat part (d) with this value of $v_3$.

**Answer:**

With $v_3$ at the bottom rail, $v_2$ is $-\frac{t}{1 \text{ms}}$ V. Similar to part (d), the voltage at the positive input is $-0.5$ V. Therefore, when $t = 0.5$ ms, $v_2 = -0.5$ V.

(f) What is $v_3$ as a function of time? Draw a graph of $v_3$ and $v_2$. Since the graph is periodic, find its period and frequency.

**Answer:**

Notice that in each of the above cases, once $v_2$ was equal to $v_+$, the output of the comparator would flip. This leads to a periodic function, where $v_3$ is either $+1$ V or $-1$ V. The period of this function is $T = 2$ ms. Notice that in each of the above cases we analyzed, we always assumed that the capacitor was initially uncharged. However, when $v_3$ switches, the capacitor will already have some charge built up on it, so it must first be drained. This is why the period is twice what we expect.

![Graph of v2 and v3](image)

(g) Suppose that we changed the value of $C_1$ to be 2 µF? What is the new period? Suppose that we change $R_5$ to be 2 kΩ. What is the new period? What if we change $R_5$ to be 0 Ω? Will this circuit still operate?

**Answer:**
Notice above we got the constant 1 ms by multiplying \( R_1 \) and \( C_1 \) together. If we double \( C_1 \), the effective period would double because it would take longer to charge \( C_1 \) to the same voltage with the same current.

Changing \( R_5 \) affects the “flip” threshold because \( v_+ \) is at a different voltage. Increasing \( R_5 \) decreases the voltage at \( v_+ \), so we would expect the flip voltage to decrease. In fact, the new period is \( \frac{4}{3} \) ms.

The circuit would not operate if \( R_5 = 0 \) Ω. The inverting input needs to be able to go above and below the non-inverting input, which is not possible if the non-inverting input is constant at the rail.

2. Correlation You are given the following two signals:

(a) Assume the two signals are periodic with period 5. Find their linear cross correlation, that is find \( \text{corr}(\vec{s}_1, \vec{s}_2) \).

**Answer:** For \( \vec{x}, \vec{y} \) that are periodic with period \( N \):

\[
\text{corr}_N(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} x[i] y[i - k]
\]

Since the signals are periodic they continue on to \(+\infty\) and \(-\infty\). We’ll start by just performing the linear correlation over one period, and ignore the signals outside of this period. Shifting the signal back will bring the next period of the signal into our range of interest. Thus we calculate the linear cross-correlation assuming the signals are periodic as below:

\[
\begin{array}{c|cccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 \\
\vec{s}_2[n] & -4 & 8 & -4 & 0 & 0 \\
\langle \vec{s}_1, \vec{s}_2[n] \rangle & -16 & + & -16 & + & 0 & + & 0 & + & 0 & = -32 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 \\
\vec{s}_2[n-1] & 0 & -4 & 8 & -4 & 0 \\
\langle \vec{s}_1, \vec{s}_2[n-1] \rangle & 0 & + & 8 & + & 0 & + & 0 & + & 0 & = 8 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 \\
\vec{s}_2[n-2] & 0 & 0 & -4 & 8 & -4 \\
\langle \vec{s}_1, \vec{s}_2[n-2] \rangle & 0 & + & 0 & + & 0 & + & 0 & + & 8 & = 8 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 \\
\vec{s}_2[n-3] & -4 & 0 & 0 & -4 & 8 \\
\langle \vec{s}_1, \vec{s}_2[n-3] \rangle & -16 & + & 0 & + & 0 & + & 0 & + & -16 & = -32 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\vec{s}_1 & 4 & -2 & 0 & 0 & -2 \\
\vec{s}_2[n-4] & 8 & -4 & 0 & 0 & -4 \\
\langle \vec{s}_1, \vec{s}_2[n-4] \rangle & 32 & + & 8 & + & 0 & + & 0 & + & 8 & = 48 \\
\end{array}
\]
Let’s continue to calculate the values of the inner product with more shifts.

| \(\bar{s}_1\) | 4 | -2 | 0 | 0 | -2 |
| \(\bar{s}_2[n-5]\) | -4 | 8 | -4 | 0 | 0 |
| \(\langle \bar{s}_1, \bar{s}_2[n-5]\rangle\) | -16 | +16 | +0 | +0 | +0 | = -32 |

| \(\bar{s}_1\) | 4 | -2 | 0 | 0 | -2 |
| \(\bar{s}_2[n-6]\) | 0 | -4 | 8 | -4 | 0 |
| \(\langle \bar{s}_1, \bar{s}_2[n-6]\rangle\) | 0 | +8 | +0 | +0 | +0 | = 8 |

| \(\bar{s}_1\) | 4 | -2 | 0 | 0 | -2 |
| \(\bar{s}_2[n-7]\) | 0 | 0 | -4 | 8 | -4 |
| \(\langle \bar{s}_1, \bar{s}_2[n-7]\rangle\) | 0 | +0 | +0 | +0 | +8 | = 8 |

| \(\bar{s}_1\) | 4 | -2 | 0 | 0 | -2 |
| \(\bar{s}_2[n-8]\) | -4 | 0 | 0 | -4 | 8 |
| \(\langle \bar{s}_1, \bar{s}_2[n-8]\rangle\) | -16 | +0 | +0 | +0 | +16 | = -32 |

| \(\bar{s}_1\) | 4 | -2 | 0 | 0 | -2 |
| \(\bar{s}_2[n-9]\) | 8 | -4 | 0 | 0 | 0 |
| \(\langle \bar{s}_1, \bar{s}_2[n-9]\rangle\) | 32 | +8 | +0 | +0 | +8 | = 48 |

Notice that the pattern repeats, this leads to the definition of circular correlation, which we will explore in the later part.

(b) Sketch the linear cross-correlation of signal 1 with signal 2, that is find : \(\text{corr}(\bar{s}_1, \bar{s}_2)\). Do not assume the signals are periodic.

**Answer:**

Represent signal 1 as the vector \(\bar{s}_1 = [0 \ 0 \ 0 \ 0 \ 4 \ -2 \ 0 \ 0 \ -2]^T\), zero-padded so that we compute only the linear correlation. Similarly, represent signal 2 as the vector \(\bar{s}_2 = [-4 \ 8 \ -4 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0\]^T\), where we once again zero pad the vector. Notice we zero pad the front of the vector \(\bar{s}_2\) but the back of the vector \(\bar{s}_1\).

The cross-correlation between two vectors is defined as follows:

\[
\text{corr}(\bar{x}, \bar{y})[k] = \sum_{i=-\infty}^{\infty} \bar{x}[i]\bar{y}[i-k]
\]

To compute the cross-correlation \(\text{corr}(\bar{s}_1, \bar{s}_2)\), we shift the vector \(\bar{s}_2\) and compute the inner product of the shifted \(\bar{s}_2\) and the vector \(\bar{s}_1\).  

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(c) Find the circular cross correlation of $\vec{s}_2$ with $\vec{s}_1$, that is find \text{circuitcorr}(\vec{s}_1, \vec{s}_2)

\textbf{Answer:}

-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4

0 \quad 20 \quad 40

Non-periodic Cross-correlation of Signals 1 and 2
Represent signal 1 as the vector $\vec{s}_1 = \begin{bmatrix} 4 & -2 & 0 & 0 & -2 \end{bmatrix}^T$. Similarly, represent signal 2 as the vector $\vec{s}_2 = \begin{bmatrix} -4 & 8 & -4 & 0 & 0 \end{bmatrix}$.

The cross-correlation between two vectors of length $N$ is defined as follows:

$$\text{circcorr}(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} \vec{x}[i] \vec{y}[(i - k)N]$$

| $\vec{s}_1$ | 4 | -2 | 0 | 0 | -2 |
| $\vec{s}_2[n]$ | -4 | 8 | -4 | 0 | 0 |
| $\langle \vec{s}_1, \vec{s}_2[n] \rangle$ | -16 | + | -16 | + | 0 | + | 0 | + | 0 | = -32

| $\vec{s}_1$ | 4 | -2 | 0 | 0 | -2 |
| $\vec{s}_2[n-1]$ | 0 | -4 | 8 | -4 | 0 |
| $\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$ | 0 | + | 8 | + | 0 | + | 0 | + | 0 | = 8

| $\vec{s}_1$ | 4 | -2 | 0 | 0 | -2 |
| $\vec{s}_2[n-2]$ | 0 | 0 | -4 | 8 | -4 |
| $\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$ | 0 | + | 0 | + | 0 | + | 0 | + | 8 | = 8

| $\vec{s}_1$ | 4 | -2 | 0 | 0 | -2 |
| $\vec{s}_2[n-3]$ | -4 | 0 | 0 | -4 | 8 |
| $\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$ | -16 | + | 0 | + | 0 | + | 0 | + | -16 | = -32

| $\vec{s}_1$ | 4 | -2 | 0 | 0 | -2 |
| $\vec{s}_2[n-4]$ | 8 | -4 | 0 | 0 | -4 |
| $\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$ | 32 | + | 8 | + | 0 | + | 0 | + | 8 | = 48

(d) Sketch the periodic autocorrelation (correlation with itself) of signal 2 assuming a period of 5.

**Answer:**

The autocorrelation is as follows. Autocorrelation is a special case of cross-correlation (it is the cross-correlation of a signal with itself). See the answer for part (c) for an example of how to compute cross-correlation.
Periodic Autocorrelation of Signal 2