1. Least Squares: A Toy Example

Let’s start off by solving a little example of least squares.

We’re given the following system of equations:

\[
\begin{bmatrix}
1 & 4 \\
3 & 8 \\
5 & 16 \\
\end{bmatrix}
\begin{bmatrix}
\vec{x}_1 \\
\vec{x}_2 \\
\end{bmatrix}
=
\begin{bmatrix}
3 \\
1 \\
9 \\
\end{bmatrix},
\]

where \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).

(a) Why can we not solve for \( \vec{x} \) exactly?

(b) Find \( \vec{x} \), the least squares estimate of \( \vec{x} \), using the formula we derived in lecture.

Reminder:

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

2. Linearizing Different Problems

Notice that least squares can only be applied to linear systems. Suppose that we have a vector \( \vec{x} \) and a vector \( \vec{y} \), and \( \vec{y}[n] = f(\vec{x}[n]) \). We would like to approximate \( f \) using least squares, where \( f \) is not necessarily a linear function.

(a) Let’s begin with a linear approximation. We want to find some \( a \) such that \( y = ax \). Set this up as a least squares problem. What are the elements in the matrix \( A \)?

(b) Let’s add a constant to the problem. Suppose that \( y = ax + b \). Set this up as a least squares problem. What are the elements in the matrix \( A \)?

(c) Suppose that \( y = ax^2 + bx + c \). Set this up as a least squares problem. What are the elements in the matrix \( A \)?

(d) Suppose that \( y = ae^{bx} \). Set this up as a least squares problem. What are the elements in the matrix \( A \)?

3. Polynomial Fitting

Let’s try an example. Say we know that the output, \( y \), is a quartic polynomial in \( x \). This means that we know that \( y \) and \( x \) are related as follows:

\[
y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4
\]

We’re also given the following observations:
(a) What are the unknowns in this question? What are we trying to solve for?
(b) Can you write an equation corresponding to the first observation \((x_0, y_0)\), in terms of \(a_0, a_1, a_2, a_3, \text{ and } a_4\)? What does this equation look like? Is it linear?
(c) Now, write a system of equations in terms of \(a_0, a_1, a_2, a_3, \text{ and } a_4\) using all of the observations.
(d) Finally, solve for \(a_0, a_1, a_2, a_3, \text{ and } a_4\) using IPython. You have now found the quartic polynomial that best fits the data!