EECS 16A   Designing Information Devices and Systems I
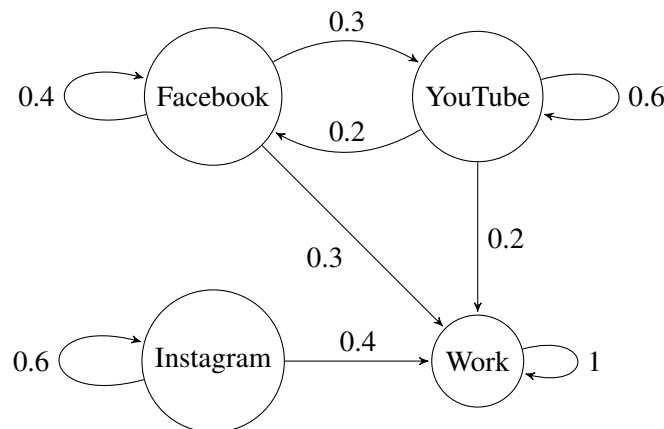Spring 2019   Discussion 2B

1. Social Media

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.

(a) What is the corresponding transition matrix?

(b) There are 150 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 70 EE16A students on Facebook, 45 on YouTube, 20 on Instagram, and 15 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

(c) If the entries in each of the column vectors of your state transition matrix summed to 1, what would this mean with respect to the students on social media? (What is the physical interpretation?)

(d) You want to predict how many students will be on each website n timesteps in the future. How would you formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 1000 timesteps in the future?

(e) **Challenging Practice Problem:** Suppose, instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void," and if a student is "leaked" from any of the other states, we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

2. Inverses

In general, the *inverse* of a matrix "undoes" the operation that the matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$
$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$
$$(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1} \qquad \text{for a nonzero scalar } k$$
$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{assuming } \mathbf{A}, \mathbf{B} \text{ are both invertible}$$

(a) Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

   i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.

   ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.

   iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?

   iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$? (*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)

   v. Are all square matrices invertible?

   vi. How can you find the inverse of a general $n \times n$ matrix?