EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 4B

Reference Definitions: Matrices and Linear (In)Dependence We've seen that the following statements are equivalent for an $n \times n$ matrix $\mathbf{A}$, meaning, if one is true then all are true:

- $\mathbf{A}$ is invertible
- The equation $\mathbf{A}\vec{x} = \vec{0}$, has a unique solution, which is $\vec{x} = \vec{0}$
- The columns of $\mathbf{A}$ are linearly independent
- For each column vector $\vec{b} \in \mathbb{R}^n$, $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution $\vec{x}$
- Null($\mathbf{A}$) $= \vec{0}$

Conversely, if one of the following is true, then all of the following are true:

- $\mathbf{A}$ is not invertible
- $\mathbf{A}\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$
- The columns of $\mathbf{A}$ are linearly dependent
- There is not be a unique $\vec{x}$ for every $\vec{b}$ where $\mathbf{A}\vec{x} = \vec{b}$
- Null($\mathbf{A}$) contains more than just the zero vector $\vec{0}$

These are part of what is known as the Invertible Matrix Theorem.

1. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

(b) $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

(c) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

(d) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

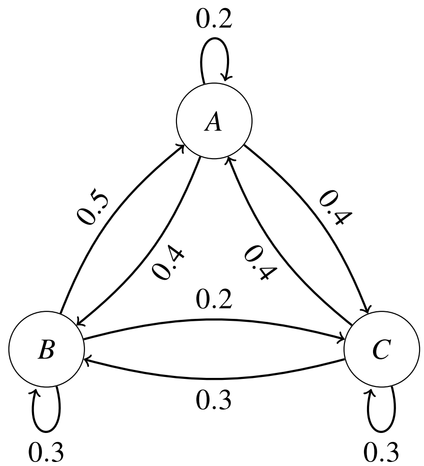

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix representing the pumps system.

(b) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Hint: If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with A_{ss}, B_{ss} and C_{ss} kiloliters of water, respectively, and apply the pumps once?

Hint II: Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.

3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$.

The following parts don't require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(c) Does a rotation matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

(d) Does a reflection matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

(e) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

(f) Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Hint: What is the rank of the matrix?

