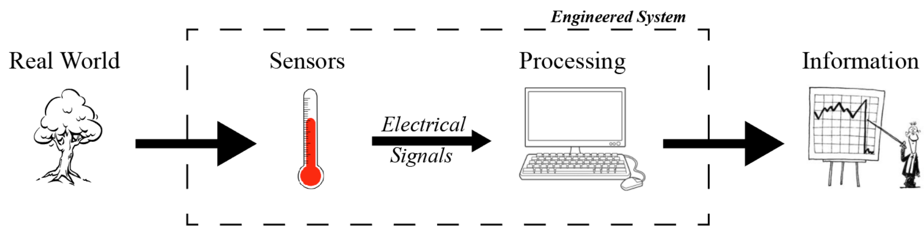
EECS 16A Designing Information Devices and Systems I
Spring 2019 Lecture Notes Note 11

11.1 Introduction to Electrical Circuit Analysis



Our ultimate goal is to design systems that solve people's problems. To do so, it's critical to understand how we go from real-world events all the way to useful information that the system might then act upon. The most common way an engineered system interfaces with the real world is by using sensors and/or actuators that are often composed of electronic circuits; these communicate via electrical signals to processing units, which are *also* composed entirely of electronic circuits. In order to fully understand and *design* a useful system, we will need to first understand **Electrical Circuit Analysis**.

In this note, we will be intentionally ignoring the underlying physics of electrical circuits, and will instead focus on a standard procedure and set of rules that will allow us to systemically "solve" such circuits, (i.e., given a circuit diagram, solve for all of the relevant electrical quantities in that circuit). The reason for this approach is that one does not need to understand any real physics in order to be able to analyze electrical circuits, and in fact, impartial or incorrect physical intuition often leads to errors and then confusion. By abstracting the physics away during the analysis step, we hope to emphasize that the analysis in and of itself is simply a matter of taking a visual diagram (representing an electrical circuit) and applying a set of rules to it that will convert the diagram in to a set of (linear) equations that can then be solved using the techniques we have developed in the first module.

11.2 Basic Circuit Quantities

Let's start with some definitions of basic quantities present in an electrical circuit. **Current** is the flow of charges (e.g. electrons) in the circuit, and **voltage** is the potential energy (per charge) between two points in the circuit. This potential energy is what causes charge to flow (ie. causes current). **Resistance** is the material's tendency to resist the flow of current.

Quantity	Symbol	Units
Voltage	V	Volts (V)
Current	I	Amperes (A)
Resistance	R	Ohms (Ω)

We use these quantities in a **Circuit Diagram**, a visual representation of how a collection of circuit elements are connected. Each circuit element has some voltage *across* it and some current *through* it.

Why is voltage "across" a circuit element? Voltage, or electric potential, is only defined *relative* to another point. A simple analogy is elevation: A mountain's summit could be 9,000 ft above sea level, but 21,000 ft above the bottom of the ocean. In both cases, the elevation is only meaningful relative to another point. For convenience, we frequently define sea level as a reference point with "0 ft of elevation" – then we can state elevation as a single number which is implicitly referenced to sea level (ex. the mountain is 9,000 ft tall). Similarly, in circuits, we will frequently define a reference point, called **ground**, against which other voltages can be measured.

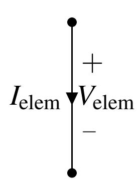
11.3 Basic Circuit Elements

How do our basic circuit quantities interact? It depends on the circuit element! For each element there is a relationship between the voltage across the element and the current through it, call an "IV Relationship." Let's look at some of the most common circuit elements and their IV relationships.

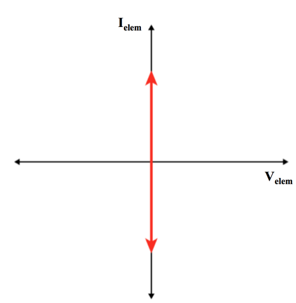
Wire: The most common element in a schematic is the wire, drawn as a solid line The IV relationship for a wire is:

$V_{elem} = 0$ A wire is an ideal connection with zero voltage across it.
$I_{elem} = ?$ The current through a wire can take any value, and is determined by the rest of the circuit.

Symbol



IV Relationship

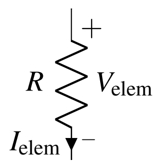


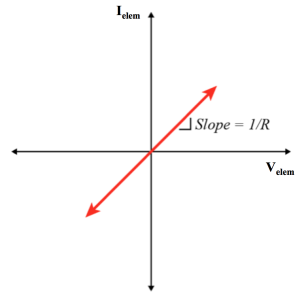
Resistor: The IV relationship of a resistor is called "Ohm's Law."

$V_{elem} = I_{elem}R$ The voltage across a resistor is determined by Ohm's Law.
$I_{elem} = \frac{V_{elem}}{R}$ The current through a resistor is determined by Ohm's Law.

IV Relationship

Symbol



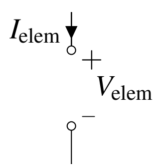


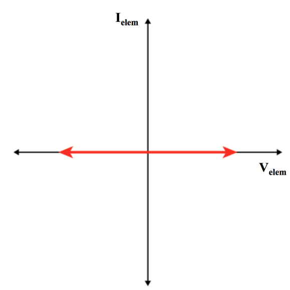
Open Circuit: This element is the dual of the wire.

$V_{elem} =$	?	The voltage across an open circuit can take any value, and is determined by the rest of the circuit.
$I_{elem} =$	0	No current is allowed to flow through an open circuit.

IV Relationship

Symbol



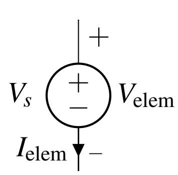


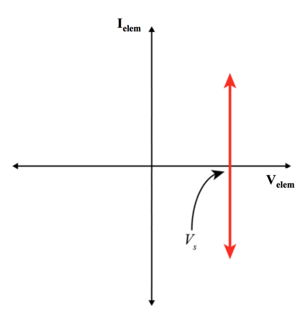
Voltage Source: A voltage source is a component that forces a specific voltage across its terminals. The $+$ and $-$ sign indicates which direction the voltage is pointing. The voltage difference between the "+" terminal and the "−" terminal is always equal to V_s, no matter what else is happening in the circuit.

$V_{elem} =$	V_s	The voltage across the voltage source is always equal to the source value.
$I_{elem} =$	?	The current through a voltage source is determined by the rest of the circuit.

IV Relationship

Symbol

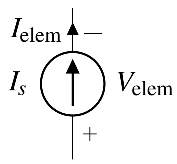




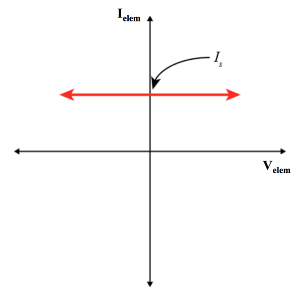
Current Source: A current source forces current in the direction specified by the arrow indicated on the schematic symbol. The current flowing through a current source is always equal to I_s, no matter what else is happening in the circuit. Note the duality between this element and the voltage source.

$V_{elem} =$	?	The voltage across a current source is determined by the rest of the circuit.
$I_{elem} =$	I_s	The current through a current source is always equal to the source value.

Symbol



IV Relationship

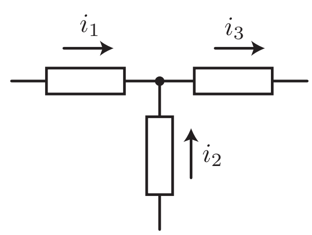


11.4 Rules for Circuit Analysis

In addition to the IV relationships for a single elements, there are also rules govern the current and voltage relationships between multiple elements.

11.4.1 Kirchhoff's Current Law (KCL)

A place in a circuit where two or more of the above circuit elements meet is called a **junction**. Kirchhoff's Current Law (KCL) states that the net current flowing out of (or equivalently, into) any junction of a circuit is identically zero. To put this more simply, the current flowing into a junction must equal the current flowing out of that junction.



For example, consider the circuit shown above. We define current flowing out of the junction to be positive, and therefore, current flowing into the junction is negative. For example, from the left branch, there is i_1 current flowing into the junction, so there is $-i_1$ current flowing out of the junction. Similarly, there is $-i_2$ current flowing out of the junction and i_3 current flowing out of the junction. Therefore, the currents must satisfy

$$(-i_1)+(-i_2)+i_3 = 0 \text{ or } i_1+i_2 = i_3 \tag{1}$$

11.4.2 Kirchhoff's Voltage Law (KVL)

Kirchhoff's Voltage Law (KVL) states that the sum of voltages across the elements connected in a loop must be equal to zero. In our elevation analogy for voltage, this is equivalent to saying"what goes up must come down". We actually will not use KVL in the analysis procedure we outline next, but it is still an important tool and can be used as an additional rule to double check your final answers.

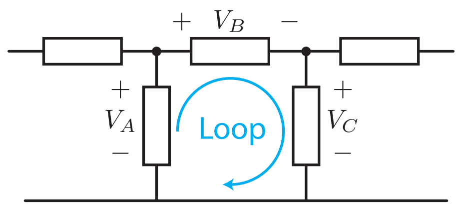


Figure 1: KVL illustration

Mathematically, KVL states that:

$$\sum_{\text{Loop}} V_k = 0. \tag{2}$$

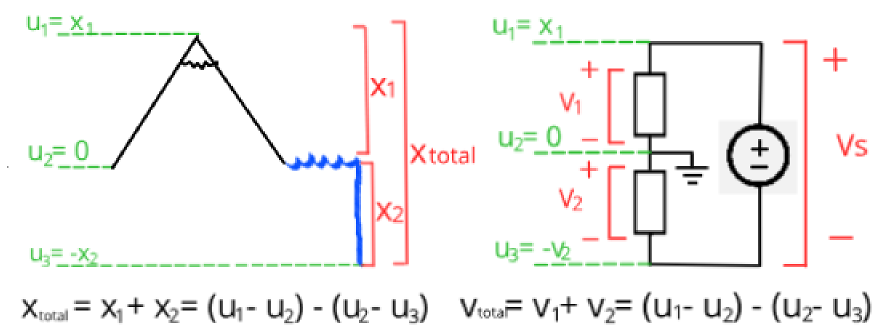
When adding the voltage "drops" around the loop we must follow a convention. If the arrow corresponding to the loop goes into the "+" of an element we subtract the voltage across that element. (In our elevation analogy, we went "downhill" from higher voltage to lower voltage so we lost "elevation.") Conversely, if the arrow goes into the "-" of an element, we add the voltage across that element (this is like going "uphill"). Following this convention for the example in Figure 1 we find:

$$V_A - V_B - V_C = 0. \tag{3}$$

Note that if we had defined the loop in the opposite direction,

$$-V_A + V_C + V_B = 0. \tag{4}$$

Thinking about an elevation analogy to voltage can help give some intuition to KVL: If you walk in a circle (a loop) so that you end up back where you started, than your total change in elevation must be zero, no matter how much you go up or down. If you walk in a line, ending up somewhere different, than your total change in elevation is equal to the sum of all of the elevation changes along the way.



11.4.3 Ohm's Law and Resistors

As already described when we introduced resistors as an element, for these elements, the voltage across them is directly proportional to the current that flows through them, where the proportionality constant is the "resistance" (R) of the device. This relationship is known as **Ohm's Law**.

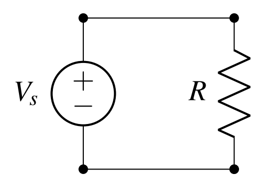
$$V_{\text{elem}} = I_{\text{elem}} R. \tag{5}$$

The unit of R is Volts/Amperes, or more commonly "Ohms" (Ω).

11.5 Circuit Analysis Algorithm

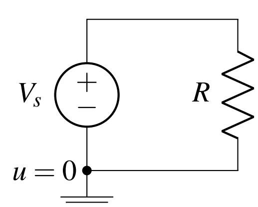
In this course, we will learn how to take a real world system and build a circuit diagram that models the behavior of that system, and we will design our own circuits for specific real world tasks. In this note, however, we will assume that we already have an accurate circuit diagram, and will learn how to analyze the circuit.

For a given circuit, we would like to find all of the voltages and currents – sometimes we call this "solving" the circuit. We'll go through an example using the following diagram, which consists of four elements: a voltage source, a resistor, and two wires. Recalling that a junction is where two or more elements meet, there are four junctions in this circuit (one at each corner of the diagram below).

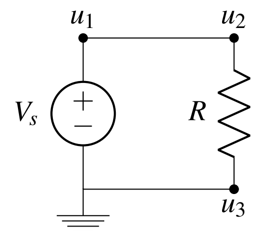


For the sake of clarity, after each step of the analysis algorithm you will be shown what the current circuit diagram would look like. When you perform the algorithm on your own, however, you do not need to redraw the circuit each time; instead you can simply label/annotate a single diagram.

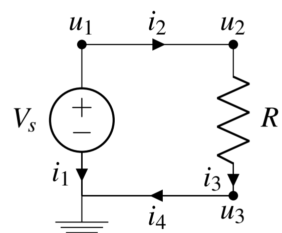
- **Step 1:** Pick a junction and label it as $u = 0$ ("ground"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.



- **Step 2:** Label all remaining junctions as some "u_i", representing the voltage at each junction relative to the zero junction/ground.

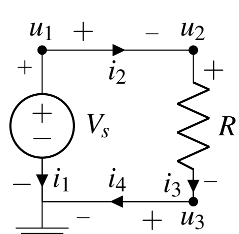


- **Step 3:** Label the current through every element in the circuit "i_n". Every element in the circuit that was listed above should have a current label, including ideal wires. (As we will describe later, once you have some familiarity with the procedure there are simplifications we can make to avoid the need to label the current in every single wire, but we describe here the most complete version of the algorithm so that you can always return to this if you ever have any doubt about whether a certain simplification is valid or not.) The direction of the arrow indidates which direction of current flow you are considering to be positive. At this stage of the algorithm, you can pick the direction of all of the current arrows *arbitrarily* - as long as you are consistent with this choice and follow the rules described in the rest of this algorithm, the math will work out correctly.

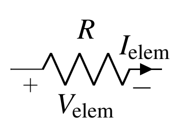
  Note that we only label the current once for each element – for example, we can label i_3 as the current leaving the resistor (as is done in the diagram) *or* we can label it as the the current entering the resistor. These are equivalent because KCL also holds within the element itself – i.e., the current that enters an element must be equal to the current that exits that same element.

- **Step 4:** Add $+/-$ labels on each element, following **Passive Sign Convention** (discussed below). These labels will indicate the direction with which voltage will be measured across that element.

Passive sign convention

The **passive sign convention** dictates that positive current should *enter* the positive terminal and *exit* the negative terminal of an element. Below is an example for a resistor:



As long as this convention is followed consistently, it does not matter which direction you arbitrarily assigned each element current to; the voltage referencing will work out to determine the correct final sign. When we discuss *power* later in the module, you will see why we call this convention "passive."

- **Step 5:** Set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the unknown circuit variables we want to solve for (currents and node potentials – that is, the i's and u's). $\mathbf{A}$ will be an $n \times n$ matrix where n is equal to the number of unknown variables. For the circuit above, we have 3 unknown potentials (u) and 4 unknown currents (i), therefore we form a 7×7 matrix.

$$\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 6:** Use KCL to fill in as many **Linearly Indpendent** rows of $\mathbf{A}$ and $\vec{b}$ as possible.

  Let's begin by writing KCL equations for every junction in the circuit.

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 + i_3 &= 0 \\ -i_3 + i_4 &= 0 \\ -i_4 - i_1 &= 0 \end{aligned}$$

  Notice the last equation we get is linearly dependent with the first three - you can see this by adding all three of the first equations to each other and multiplying the entire result by -1. In order to end up with a square and invertible $\mathbf{A}$ matrix, we will therefore omit this equation. Note that in general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; skipping the junction that has been labeled as ground is a common choice.

  Now we put these equations in matrix form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 7:** Use the IV relationships of each of the elements to fill in the remaining equations (rows of $\mathbf{A}$ and values of $\vec{b}$).

  In this example, we need four more linearly independent equations, and there are four circuit elements, each with their own IV relationship (this is not a coincidence, as will be explained shortly). We use what we know about each element to form four more equations.

  We know that the difference in potentials across the voltage source must be the voltage on the voltage source. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law. For the wires, we know the difference in potential is 0. Thus, we have the following equations:

$$\begin{aligned} u_1 - 0 &= V_s \\ u_1 - u_2 &= 0 \\ u_2 - u_3 &= Ri_3 \\ u_3 - 0 &= 0 \end{aligned}$$

After filling in these equations, our matrix is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -R & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

At this point the analysis procedure is effectively complete - all that's left to do is solve the system of linear equations (by applying Gaussian Elimination, inverting $\mathbf{A}$, etc.) to find the values for the u's and i's.

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -V_s/R \\ V_s/R \\ V_s/R \\ V_s/R \\ V_s \\ V_s \\ 0 \end{bmatrix}$$

Before we move on, it is worth pausing at this point to highlight why the procedure always works, and in particular, why we will always have as many equations as we do unknowns. If a circuit has m elements in it and n junctions, there will be $(n-1)$ u's (since we have defined one of them as ground/zero), and m currents (one for each element). Since each element has a defining I-V relationship, Step 7 will provide us with m equations. Similarly, with n junctions, we will get $(n-1)$ linearly independent KCL equations from Step 6. As a side note, the order of Steps 6 and 7 can be interchanged - in fact, you may want to do Step 7 (IV relations) first since when you then come back to Step 6 (KCL) you just have to fill in as many equations using KCL as needed to make the $\mathbf{A}$ matrix square.

11.6 Simplifying the Circuit Analysis Procedure

While the analysis procedure we described in the previous section will always work, and introducing the procedure at this level of comprehensiveness is necessary to ensure that one can always follow it successfully, as is most likely clear, even for very simple circuits the procedure will quickly involve a large number of variables and hence large matrices. Fortunately, we can substantially reduce the number of variables by noticing two things:

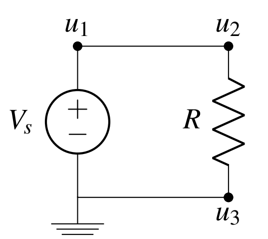
1. There is no voltage drop across wires. Therefore, the node potentials at two end of a wire are always equal.
2. When a junction involves only two elements, KCL tells us that the current flowing in through the first element must equal the current flowing out through the second element.

The next two sections describe in more detail how we can use these observations to simplify solving a circuit.

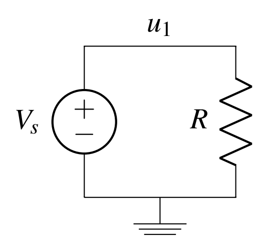
11.6.1 Labeling Nodes Instead of Junctions

Since wires always have zero voltage drop across them, there is no specific need for us to keep track of the voltage (relative to ground) on the two sides of a wire separately. In other words, **all of the junctions that are connected to each other by wires can be labeled with a single voltage variable u.** A set of such junctions connected to each other only via wires is defined as a **node**. (Formally, a node is defined as a region of the circuit that is "equipotential" - i.e., that has no voltage drop across it - but since there is no voltage drop across wires, this is exactly the same as our earlier criteria.)

As an example, let's consider the circuit we were analyzing, but return to Steps 1 and 2. As shown below, the junctions previously labeled as u_3 and ground are connected by a wire and are therefore a single node. We can label that entire node as ground. Similarly, the junctions previously labeled as u_1 and u_2 are also connected by a wire, so are also a single node. We can label that entire node as u_1.



Original procedure:
Labeling junctions



Simplified procedure:
Labeling nodes

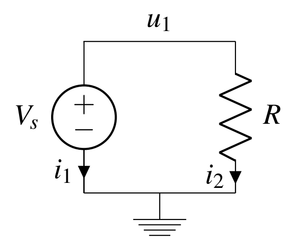
When we followed the original analysis procedure where we labeled *junctions*, we ended up with three unknown u's; by labeling only the *nodes*, we have simplified down to a single unknown u (u_1). In general, since wires are abundant in circuit diagrams, labeling only the nodes (instead of the junctions) will substantially reduce the number of variables. At the end of this note, we include a more thorough procedure for identifying all of the nodes in a circuit.

11.6.2 Trivial Junctions

We define a **trival junction** to be a junction connecting only two elements. KCL dictates that the current entering the junction must be equal to the current exiting. Since there are only two elements, it follows that the two currents must be equal (as long as we label the direction of current flow to be the same – if not, the currents will simply be opposite in sign).

Therefore, another simplification to our analysis procedure is to **label the currents only in the non-wire elements** in our circuit. (Sometimes these currents are called *branch currents*). We can later find the current in any given wire by looking for a trivial junction between the wire and a non-wire element. When we use KCL, we can now consider *nodes* (instead of junctions) — i.e. the current flowing into the node is equal to the current leaving the node.

Returning to our example, if we repeat Step 3 (and assume labeled nodes rather than junctions, as explained in the previous section), we would now label only the current through the two non-wire elements: the voltage source and the resistor.



With this simplified approach, when we get to Step 6 (KCL), we would apply KCL at the node u_1, which would result in the equation:

$$-i_1 - i_2 = 0 \tag{6}$$

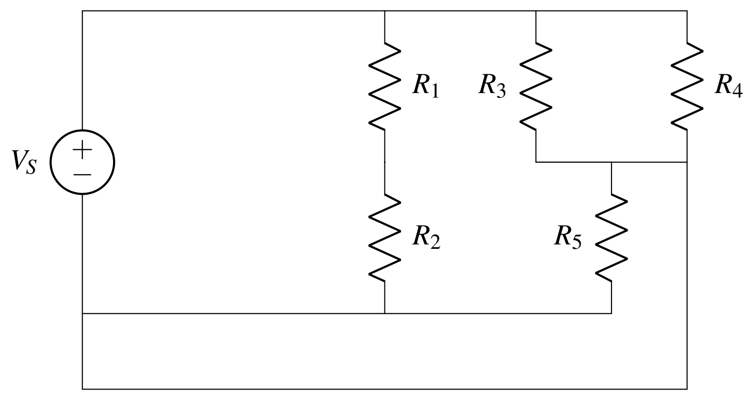
11.6.3 Summary of Simplified Procedure

By labeling nodes instead of junctions and labeling currents in non-wire elements only, we can greatly reduce the number of variables in our circuit analysis procedure, so this is what we will do in the future. Here's a summary of the steps:

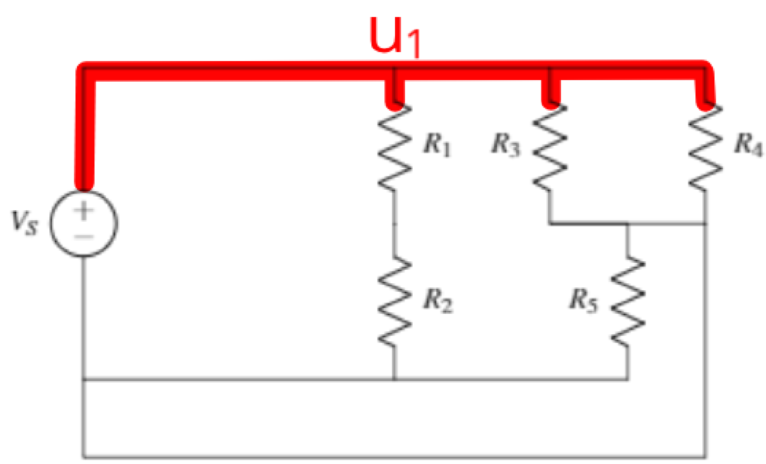
- **Step 1:** Pick a **node** and label it as $u = 0$ ("ground"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.
- **Step 2:** Label all remaining **nodes** as some "u_i", representing the voltage at each node relative to the ground node.
- **Step 3:** Label the current through every **non-wire** element in the circuit "i_n".
- **Step 4:** Add $+/-$ labels (indicating direction of voltage measurement) on each **non-wire** element by following the passive sign convention.
- **Step 5:** Set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the u_i's and i_n's defined in the previous steps.
- **Step 6:** If there are n **nodes** (including the ground node), use KCL on $(n-1)$ nodes to fill in $(n-1)$ rows of $\mathbf{A}$ and $\vec{b}$.
- **Step 7:** If there m **non-wire** elements, use the IV relationships of each non-wire element to fill in the remaining m equations (rows of $\mathbf{A}$ and values of $\vec{b}$).
- **Solve with your favorite technique from linear algebra!**

11.7 Guide to Finding Nodes

Our simplified circuit analysis procedure is much simpler than the full version because there are fewer variables, and labeling nodes instead of junctions is one of the main reasons. Here, we will go over a method you can use to correctly identify all of the nodes in a circuit. We'll go through this methods while applying it to an example circuit, shown below.

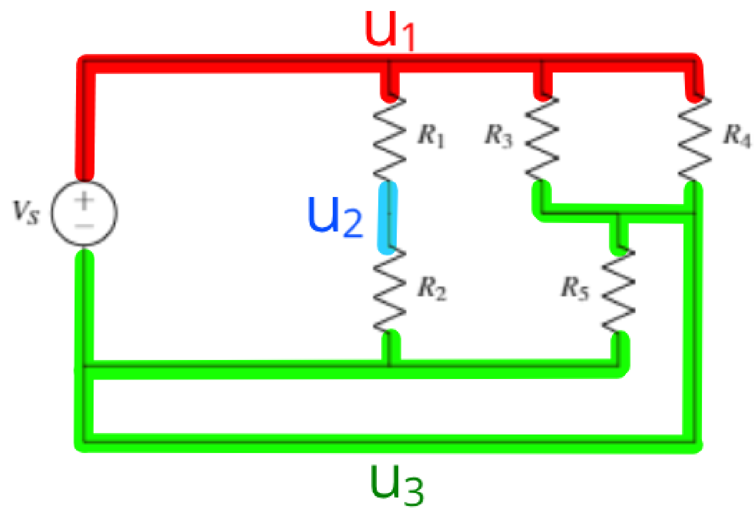


We'll find nodes one at a time with this method (the order we find the nodes is arbitrary). We start by choosing a color to represent the first node (red, in this example). Then we choose a starting point on the circuit,such as the upper left corner. From this point, we trace (in red) along all of the connected wires until we hit a non-wire component. Everything traced in red is part of a single node.



Then we choose a new color and a new point on the circuit *that is not already colored*. We repeat the process: tracing over all wires that are connected to the this new point and stop when a non-wire element is reached. Repeat this process for new colors until all of the wires have been identified.

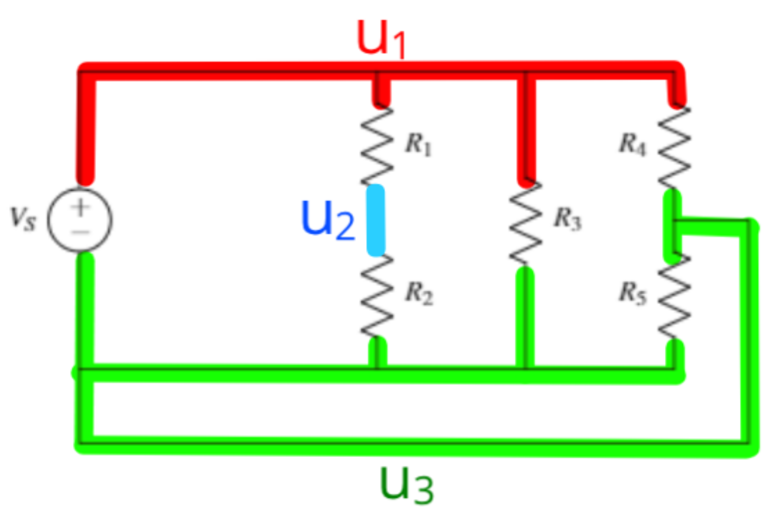
We've now identified all of the nodes! There are three in this example circuit. Note that one of these would be labeled "ground" in the seven step circuit analysis procedure, so we would only have to solve for two unknown node potentials.



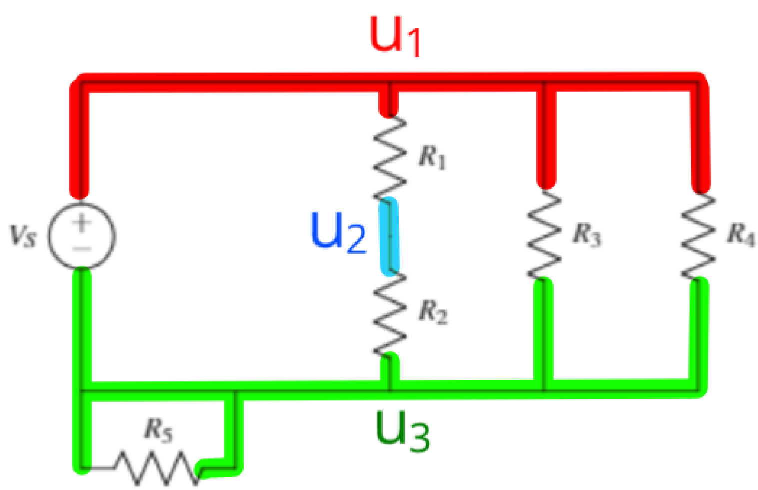
Once we've identified the nodes, we can use this knowledge to help us redraw the circuit in a way that the currents and voltages in each element don't change. In other words, we want to use a different diagram to represent the same circuit behavior. This is useful because sometimes it is easier to see patterns in a circuit diagram if it's draw differently. However, we must be careful to not change the circuit when we redraw it.

If we don't want the circuit to change when we redraw it, **each non-wire component must be connected to the same nodes on either end.** This is because the voltage drop over each element is dependent only on the nodes it is connected to, and the current through each element is determined by the voltage drop and the IV relationship of that element.

For example, we can redraw our example circuit with R_3 in a different location, as long as one end of R_3 is still connected to the red node (u_1) and the other end of R_3 is still connected to the green node (u_3):



We can similarly rearrange other components if needed. In this case, we move R_5 but maintain that it is connected to the green node, u_3, on both sides. Note that all of the elements have the same node connections as they did in the original circuit. Therefore this circuit will have the exact same behavior as the original:



Sometimes redrawing a circuit can make it easier to analyze. However, it is important to stay consistent with node labeling and make sure the redrawn circuit is still the same as the original.

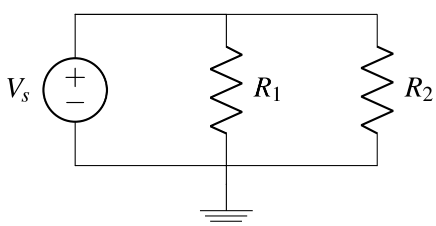
11.8 Practice Problems

These practice problems are also available in an interactive form on the course website (http://ee16a.com/hw-practice#/).

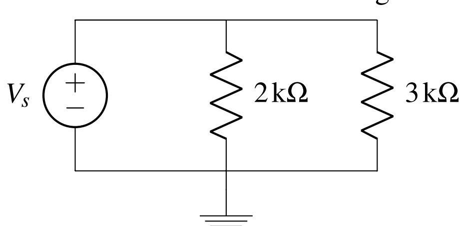
1. True or False: A voltage source can have any current through it.

2. True or False: A current source can have any voltage across it.

3. True or False: The voltage across R_1 and across R_2 is the same.

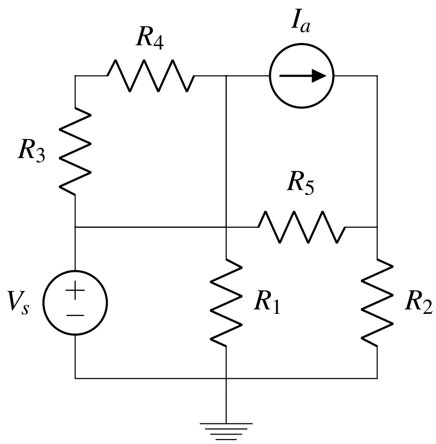


4. True or False: The current through the resistors is the same.

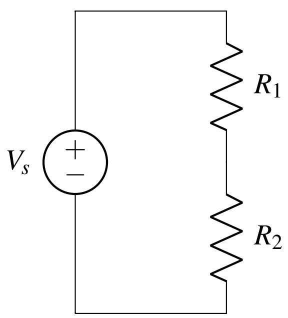


5. If you have n nodes in a circuit with k non-wire elements connecting the nodes, how many equations do you need to solve for all node potentials and element currents? Remember that one node needs to be grounded.

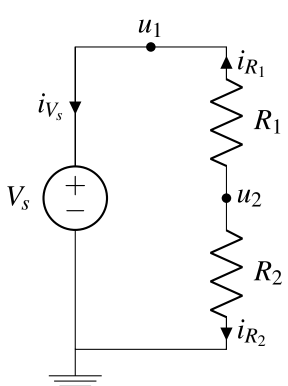
6. How many nodes would you need to label to perform nodal analysis? Include nodes for ground and for V_s.



7. How many nodes are in the following circuit?

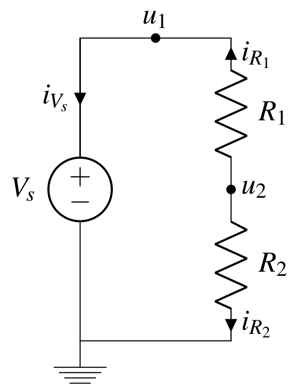


8. Assume that you have picked the ground node and labeled the node potentials and branch currents as follows.



What are the $+/-$ labels for R_1 and R_2 according to passive sign convention?

9. For the same circuit as above, formulate a system of equations to solve for all node potentials and branch currents.


u_1
i_{R_1}
i_{V_s}
R_1
V_s
u_2
R_2
i_{R_2}