EECS 16A Designing Information Devices and Systems I
Spring 2019 Lecture Notes Note 5

5.1 Interpretation: Water Reservoirs and Pumps

One way of visualizing matrix-vector multiplication is by considering water reservoirs and water pumps. We are presenting this example because it is vital as an engineer that you understand the ideas that we are talking about in an intuitive way, and this intuition often comes from having a series of examples that make sense. After all, we define mathematical operations the way that we do because these definitions are useful; they don't come out of nowhere. The act of doing mathematics and engineering is often about making definitions and seeing where they lead us, while checking the consistency of these definitions with what we are trying to model in the real world.

For these examples, we will have three water reservoirs, A, B, C. Let's say the initial amounts of water they respectively hold are A_0, B_0, C_0. Next, say we have a system of pumps connecting the reservoirs that move certain amounts of water between the reservoirs every day. We can represent the reservoirs as the following vector, where each element describes how much water is currently in that reservoir:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

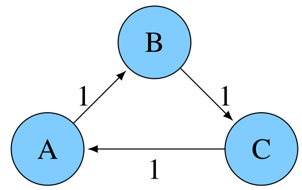
Then, we can represent the system of pumps as a matrix:

$$\begin{bmatrix} P_{A\to A} & P_{B\to A} & P_{C\to A} \\ P_{A\to B} & P_{B\to B} & P_{C\to B} \\ P_{A\to C} & P_{B\to C} & P_{C\to C} \end{bmatrix}$$

Each element $P_{i\to j}$ represents the fraction of water in reservoir i that goes into reservoir j the next day. The matrix acts on the vector just as the pumps act on the reservoirs, performing a transformation — multiplying a vector representing the distribution of water in one day by the pump matrix will give a vector with the distribution of water the next day. We call this matrix a **state transition matrix**. This example can also extend to matrix-matrix multiplication. Both this pumps and reservoirs example and a similar example (PageRank — how search engines can use link information to figure out which pages are important) will show further applications of linear algebra.

5.1.1 Basic Pump

The most basic pump system will move all water from one reservoir into another. Pictorially, we can show this as follows (blue circles are the reservoirs and arrows represent how the pumps move the water):



The corresponding matrix-vector multiplication is:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

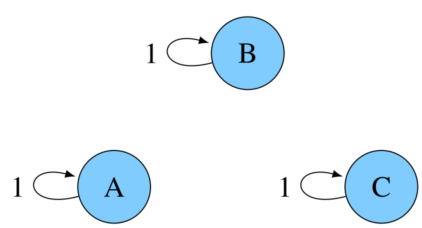
Each time the pumps act on the reservoirs, all of the water in reservoir A flows into reservoir B. All of the water in reservoir B flows into reservoir C. All of the water in reservoir C flows into reservoir A. If A, B, and C all start with the same amount of water, then the pumps acting on the reservoirs would not change the amount of water in each reservoir. As an example, let the amount of water in each reservoir initially be A_0, B_0, C_0. Then we can calculate the amount of water in each reservoir after activating the pumps once (A_1, B_1, C_1) as follows:

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} C_0 \\ A_0 \\ B_0 \end{bmatrix}$$

5.1.2 Identity Matrix Pump

What happens when your pump system can be represented as the identity matrix? What does that mean? If the initial amounts of water in the reservoirs are represented by the vector $\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix}$, and the identity matrix represents how the pumps move the water, after one activation of the pumps, nothing changes!

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix}$$



5.1.3 Drain

Another special matrix is the zero matrix:

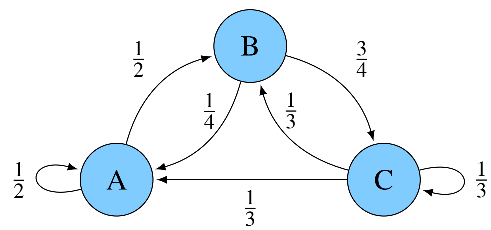
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In terms of the reservoirs, it would be some sort of drain (e.g. an evil monster evaporated all of the water in the three reservoirs). The zero matrix acting on the reservoirs results in zero water left in each reservoir.

This does not obey water conservation — the total amount of water after the pump matrix is applied will not equal the initial total — but can still be represented as a matrix.

5.1.4 Conservation of Water

Now let's look at what happens when pumps move different amounts of water from each reservoir into other reservoirs. Specifically, let's work with this diagram:



Now, let us describe these pumps with this matrix:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}$$

Each element of the matrix still represents a pump and indicates how much water is moved where. The first row indicates how much of each reservoir contributes to reservoir A when the pumps are activated. The second row does the same for reservoir B, and the third row is for reservoir C. For example, the upper left element $\frac{1}{2}$ tells us that half of what is in reservoir A will stay in reservoir A. Similarly, the $\frac{1}{2}$ on the middle left tells us that the other half of what was in reservoir A will flow into reservoir B when the pumps turn on. As we can see, each column of the matrix sums to one. This means the water is conserved (none is mysteriously lost or gained). The water will either stay in the original reservoir or move to a different one.

This is a useful fact about water moving between pools with no evaporation, but it is not something that is going to hold in all useful applications of matrices.

After activating the pumps once, how do we know how much water is in each reservoir? That can be calculated with a matrix-vector product, just as we saw with the simpler pump models:

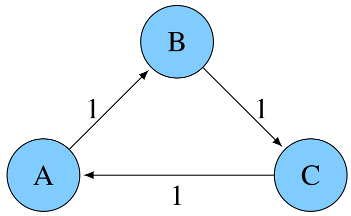
$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot A_0 + \frac{1}{4} \cdot B_0 + \frac{1}{3} \cdot C_0 \\ \frac{1}{2} \cdot A_0 + 0 \cdot B_0 + \frac{1}{3} \cdot C_0 \\ 0 \cdot A_0 + \frac{3}{4} \cdot B_0 + \frac{1}{3} \cdot C_0 \end{bmatrix}$$

5.1.5 Matrix Multiplication Examples

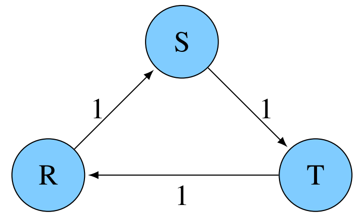
Now let's look at how matrix-matrix multiplication can be applied to water reservoirs and pumps.

5.1.6 Twin Cities

In this scenario, we have two cities that each have three reservoirs. The pump systems in the cities are identical. Let's start with the basic pump system.



(a) City 1



(b) City 2

The pump system can still be represented as a matrix, like before:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

However, now the pump system acts on two sets of reservoirs instead of just one. Can our pump matrix act on two vectors representing the reservoirs instead of just one? We can combine the two vectors representing the water reservoirs in each city into a single matrix, with a column for each reservoir vector as follows:

$$\begin{bmatrix} A & R \\ B & S \\ C & T \end{bmatrix}$$

Now, let's use the pump matrix to find the water distribution in both cities in a single calculation. In City 1, the reservoirs initially have water amounts A_0, B_0, C_0. In City 2, the reservoirs initially have water amounts R_0, S_0, T_0. Once the pumps act on the reservoirs, the amount of water in each reservoir can be found through matrix-matrix multiplication:

$$\begin{bmatrix} A_1 & R_1 \\ B_1 & S_1 \\ C_1 & T_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 & R_0 \\ B_0 & S_0 \\ C_0 & T_0 \end{bmatrix} = \begin{bmatrix} C_0 & T_0 \\ A_0 & R_0 \\ B_0 & S_0 \end{bmatrix}$$

If the cities have identical but more complicated pumps (such as the conservation pumps in the previous example), finding out how the reservoirs change is the same process. All that would be different is the "pump system" matrix.

What happens if you have the same pump-reservoir system in k cities? To find out how the pumps act on the reservoirs, you can still use matrix-matrix multiplication. One matrix describes the pumps, while the other describes the reservoirs. There would be k columns in the reservoir matrix, because each column is a vector that represents the reservoirs of a certain city.

In this case, we see a matrix acting on another matrix to transform multiple vectors the same way. Because of this, we can also see why the dimensions of the matrix have certain restrictions. The number of columns in the pumps matrix must match the number of rows in the reservoir matrix. The pumps matrix acts on each column of the reservoir matrix to produce a new column for the resulting matrix that describes amount of water for that city's reservoirs.

5.1.7 Activate Pumps Once... And Then Once More

Now, imagine we have one system of pumps for one city with three reservoirs. How can we calculate the amount of water in each reservoir after activating the pumps twice? From matrix-vector multiplication, we know how to find the amount after one activation. If matrix A represents the pumps and $\vec{v}_0$ represents the initial reservoir vector, then $\vec{v}_1 = A \cdot \vec{v}_0$ will tell us how much water is in each reservoir after one activation. Then $\vec{v}_2 = A \cdot \vec{v}_1$ will tell us how much water is in each reservoir after the second activation.

But from the reservoirs' standpoints, how they got from $\vec{v}_0$ to $\vec{v}_2$ does not matter. For all they know, it could have been some other system of pumps (matrix B) that acted on the initial reservoir vector ($\vec{v}_0$) that resulted in $\vec{v}_2$. This means that one set of pumps acting twice on the reservoirs is equivalent to *another* matrix acting on the reservoirs:

$$\begin{aligned} A\vec{v}_1 &= \vec{v}_2 \\ A(A \cdot \vec{v}_0) &= B \cdot \vec{v}_0 \\ (A \cdot A)\vec{v}_0 &= B \cdot \vec{v}_0 \\ A^2 &= B \end{aligned}$$

As an example, let's take the pump system from the conservation example in section 5.1.4:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}, \vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let's calculate $\vec{v}_2$:

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{13}{12} \\ \frac{5}{6} \\ \frac{13}{12} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{13}{12} \\ \frac{5}{6} \\ \frac{13}{12} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{65}{72} \\ \frac{71}{72} \end{bmatrix}$$

For comparison:

$$B = A^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{13}{36} \\ \frac{1}{4} & \frac{3}{8} & \frac{5}{18} \\ \frac{3}{8} & \frac{1}{4} & \frac{13}{36} \end{bmatrix}$$

$$B \cdot \vec{v}_0 = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{13}{36} \\ \frac{1}{4} & \frac{3}{8} & \frac{5}{18} \\ \frac{3}{8} & \frac{1}{4} & \frac{13}{36} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{65}{72} \\ \frac{71}{72} \end{bmatrix}$$

From this example, we can see that matrix-matrix multiplication results in an equivalent matrix. Pump system A acting twice on the reservoirs is the same as pump system B acting once on the reservoirs.

5.1.8 A Multitude of Pumps

Another example of matrix-matrix multiplication with these pumps and reservoirs is when two (or more) *different* sets of pumps act sequentially on a city's reservoirs. From the previous example, we know that a matrix multiplied by another matrix is equivalent to another matrix. That principle can be applied here. So if we have Pump System A act on the reservoirs ($\vec{v}_0$) and then Pump System B act on the reservoirs, it is the same as if some other Pump System C acted on the reservoirs:

$$\begin{aligned} B \cdot (A \cdot \vec{v}_0) &= C \cdot \vec{v}_0 \\ (B \cdot A)\vec{v}_0 &= C \cdot \vec{v}_0 \\ B \cdot A &= C \end{aligned}$$

5.1.9 Continuous vs. Discrete Pumps

In all of the examples above, we've assumed that the pumps act instantaneously. That is, each time the pumps transfer water, first we calculate how much water will be moved based on the initial water levels in each reservoir. Then, all of the water is moved instantaneously. (At any given time, water entering reservoir A is not used in calculating how much water leaves reservoir A.) We can repeat this process (every day, minute, hour, etc), but each time all the water moves instantaneously. We call this process a *discrete time system* because water is transferred only at a discrete times.

While some physical properties happen in discrete time, others happen in *continuous time* – in other words, not instantaneously. We can describe continuous time systems with *differential equations*, which you will learn more about in EE 16B. But even without these techniques, we can still approximate the solution to a continuous time system by modeling it as a discrete time system where we take very small steps in time. Here, small is relative to how long the process takes: if it takes a minute for the pumps to transfer water, we could calculate the new water levels every second. If we are trying to model how light travels in space, we might need to calculate a new time step every femptosecond!

5.2 Practice Problems

These practice problems are also available in an interactive form on the course website.

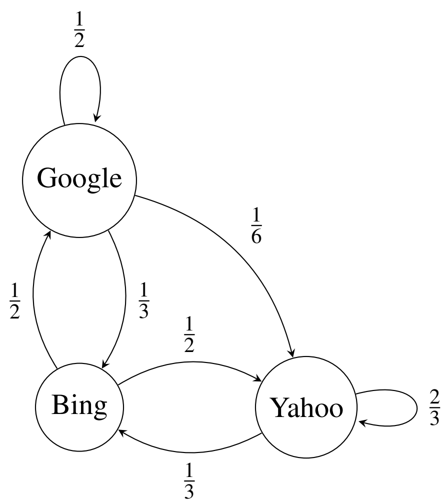
1. Let the state transition matrix $\begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0.5 & 1 \\ 0.4 & 0.2 & 0 \end{bmatrix}$ represent people moving between three cities. If $\vec{x}[0] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}$, find $\vec{x}[1]$.

2. Let the state transition matrix $\begin{bmatrix} 0.5 & 0.3 & 0 \\ 0 & 0.5 & 1 \\ 0.4 & 0.2 & 0 \end{bmatrix}$ represent people moving between three cities. Do people stay within these three cities?

3. Let the state transition matrix $\begin{bmatrix} 0.1 & 0.1 & 0.4 & 0.5 \\ 0.6 & 0.15 & 0 & 0.2 \\ 0.3 & 0.5 & 0.3 & 0.2 \\ 0 & 0.25 & 0.3 & 0.1 \end{bmatrix}$ represent the transfer of water between different buckets. The amount of water in each bucket a, b, c, and d at time n is $\begin{bmatrix} 3 \\ 4 \\ 19 \\ 1 \end{bmatrix}$. How much water is

in bucket c at time $n+1$?

4. Consider the web traffic among the search engines given below. Write the state transition matrix for this system assuming that the state vector is $\vec{x}[t] = \begin{bmatrix} x_{\text{Google}}[t] \\ x_{\text{Yahoo}}[t] \\ x_{\text{Bing}}[t] \end{bmatrix}$.



5. Is the web traffic system modeled in the previous question conservative, i.e., is the number of web surfers in the system constant?

6. If a column adds up to a number larger than 1, what does this imply about the corresponding node?

   (a) People are leaving the system at that node.

   (b) People are entering the system at that node.

   (c) The node can exist in a conservative system.

   (d) The node has been wrongly modeled in the system.