1. Lateration with Linear Systems of Equations (Spring 2017 Final)

In this problem, we’ll set up the system of equations you need in order to uniquely determine the microphone position while only knowing the relative time differences of arrival. We will use 5 beacons for the setup on this exam.

All beacons transmit their unique signal at time 0 that each arrive at the microphone at different time \( T_m \) depending on the microphone position. Let \( \tau_m = T_m - T_0 \) be the time difference of arrival between beacon \( m \) and beacon 0.

(a) Using the fact that \( R_m = vT_m \), where \( R_m \) is the distance of the microphone from beacon \( m \) and \( v \) is the speed of the audio signals in air, show that we can write:

\[
0 = v\tau_m + 2R_0 + \frac{R_0^2 - R_m^2}{v\tau_m},
\]

where \( \tau_m \) is the time difference of arrival for beacon 0 and beacon \( m \).

**Answer:**

We can write:

\[ v\tau_m = R_m - R_0 \]

Next, we re-arrange and square to get:

\[
R_m^2 = (v\tau_m + R_0)^2
\]

\[
R_m^2 = (v\tau_m)^2 + 2(v\tau_m)R_0 + R_0^2 - R_m^2
\]

\[
0 = (v\tau_m)^2 + 2(v\tau_m)R_0 + R_0^2 - R_m^2
\]

\[
0 = v\tau_m + 2R_0 + \frac{R_0^2 - R_m^2}{v\tau_m}
\]

(b) We introduce coordinates based on beacon 0. Beacon 0 is located at \((0,0)\), beacon \( m \) is located at known positions \((x_m, y_m)\). We try to find the microphone position in the sensor plane, given by \((x,y)\). We see that \( R_m \), the distance of the microphone from beacon \( m \), is a function of the microphone position:

\[
R_m = \sqrt{(x-x_m)^2 + (y-y_m)^2}
\]

By plugging the relationship from Equation (2) into Equation (1) above for \( R_0 \), \( R_1 \), and \( R_2 \), we can write a system of equations with two equations and two unknowns:

\[
0 = v\tau_1 + 2\frac{x_1x - x_1^2 + 2y_1y - y_1^2}{v\tau_1}
\]

\[
0 = v\tau_2 + 2\frac{x_2x - x_2^2 + 2y_2y - y_2^2}{v\tau_2}
\]
What is the minimum number of beacons you would need to use to create this system of equations? Justify why you cannot use linear algebra methods to solve the system of equations in this part.

**Answer:**

In order to create this system of equations, we would need 3 beacons: beacon 0 and 1 to generate coefficients in the first equation, and beacon 0 and 2 to generate coefficients in the second equation. As written, the system of equations requires us to evaluate the square root of the sum of the squares of our unknowns. This operation cannot be expressed by in a linear form. In effect, \( \sqrt{x^2 + y^2} \) cannot be written as \( k_1x + k_2y + k_c \).

This needs to be true in order to write a matrix equation, \( A\vec{x} = \vec{b} \) where \( \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \).

(c) We can use extra information to make the system linear. Subtract Equation 3 from Equation 4. This is a linear equation and should have the form of

\[ ax + by + c = 0. \] (5)

How many linear equations of this form can we construct with 3 beacons, and how many unknowns do we have? Can you use this system to find a unique solution for \((x, y)\) using linear algebra methods? Why or why not?

**Answer:**

We can start with the equations given in the previous parts.

\[ 0 = v\tau_1 + 2\sqrt{x^2 + y^2} + \frac{2x_1x - x_1^2 + 2y_1y - y_1^2}{v\tau_1} \]
\[ 0 = v\tau_2 + 2\sqrt{x^2 + y^2} + \frac{2x_2x - x_2^2 + 2y_2y - y_2^2}{v\tau_2} \]
\[ 0 = v\tau_2 - v\tau_1 + 2\sqrt{x^2 + y^2} - 2\sqrt{x^2 + y^2} + \frac{2x_2x - x_2^2 + 2y_2y - y_2^2}{v\tau_2} - \frac{2x_1x - x_1^2 + 2y_1y - y_1^2}{v\tau_1} \]
\[ 0 = \left( \frac{2x_2}{v\tau_2} - \frac{2x_1}{v\tau_1} \right)x + \left( \frac{2y_2}{v\tau_2} - \frac{2y_1}{v\tau_1} \right)y + v\tau_2 - v\tau_1 + \frac{x_1^2 + y_1^2}{v\tau_1} - \frac{x_2^2 + y_2^2}{v\tau_2} \]

\[ a = \frac{2x_2}{v\tau_2} - \frac{2x_1}{v\tau_1} \]
\[ b = \frac{2y_2}{v\tau_2} - \frac{2y_1}{v\tau_1} \]
\[ c = v\tau_2 - v\tau_1 + \frac{x_1^2 + y_1^2}{v\tau_1} - \frac{x_2^2 + y_2^2}{v\tau_2} \]

With three beacons, we can only write one linear equation, but we have two unknowns.

\[ ax + by + c = 0 \]

Therefore, we cannot find a unique solution for \( x \) and \( y \), since all solutions along the line described by this equation are valid.
(d) For our microphone lateration system, we want to use 5 beacons. Given 5 beacons, how many unique
equations of the form \( ax + by + c = 0 \) can you write? (You do not need to write out the expressions
for \( a, b, \) and \( c.\) ) Write a matrix equation, \( \mathbf{A}\mathbf{x} = \mathbf{b} \) in terms of \( a_n, b_n, \) and \( c_n, \) where the subscript \( n \)
corresponds to one of the unique equations. Also, use \( \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \)

**Answer:**

With 5 total beacons, we can write 3 unique sets of coefficients, \( a_n, b_n, \) and \( c_n.\) Therefore, the setup of \( \mathbf{A} \) and \( \mathbf{b} \) will be:

\[
\mathbf{A} = \begin{bmatrix}
    a_1 & b_1 \\
    a_2 & b_2 \\
    a_3 & b_3 \\
\end{bmatrix}
\]

\[
\mathbf{b} = \begin{bmatrix}
    -c_1 \\
    -c_2 \\
    -c_3 \\
\end{bmatrix}
\]

Written out,

\[
\begin{bmatrix}
    a_1 & b_1 \\
    a_2 & b_2 \\
    a_3 & b_3 \\
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
\end{bmatrix} = \mathbf{b} = \begin{bmatrix}
    -c_1 \\
    -c_2 \\
    -c_3 \\
\end{bmatrix}
\]