1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:
• An $n \times d$ dimension signature matrix $S$ with columns $S_i$
• An $n$ dimension measurement vector $\vec{y}$
• The sparsity level $m$ of the signal

Outputs:
• An estimate $\hat{\vec{x}}$ in $\mathbb{R}^d$ of the ideal message
• A set $\Lambda_m$ containing $m$ elements from $\{1, 2, \ldots, d\}$
• An $n$-dimensional approximation $\hat{\vec{y}}$ of the measurement vector $\vec{y}$
• An $n$-dimensional residual $\vec{r} = \vec{y} - \hat{\vec{y}}$

1: procedure OMP($S$, $\vec{y}$, $m$)
2: $\vec{r} \leftarrow \vec{y}$ (portion of measurement to be explained)
3: $A_0 \leftarrow []$ (signatures found in the measurement)
4: $j \leftarrow 1$ (iteration variable)
5: $\Lambda_0 \leftarrow []$ (indices of vectors found)
6: while $\vec{r} \neq \vec{0}$ do
7:    procedure FIND SIGNATURE($\vec{r}$, $S$)
8:        $k \leftarrow \text{argmax}_i \langle \vec{S}_i, \vec{r} \rangle$
9:        $A_j \leftarrow [A_{j-1} \quad \vec{S}_k]$
10:       $\Lambda_j \leftarrow \Lambda_{j-1} \cup k$
11:    end procedure
12:    procedure FIND PROJECTION($A_j$, $\vec{y}$)
13:        $\hat{x}_j = (A_j^T A_j)^{-1} A_j^T \vec{y}$
14:        $\hat{y}_j = A_j \hat{x}_j$
15:        $\vec{r} \leftarrow \vec{y} - \hat{\vec{y}}$
16:    end procedure
17:    $j \leftarrow j + 1$
18: end while
19: The estimate for the ideal signal has non-zero indices at the components listed in $\Lambda_m$. The value of $\hat{x}$ in component $\lambda_j$ equals the $j$th component of $\hat{x}$.
20: end procedure

2. One Magical Procedure (Fall 2015 Final)

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix $M$ defined by column vectors
\(\vec{c}_1, \ldots, \vec{c}_5\), such that:

\[
M \vec{x} = \begin{bmatrix}
\vec{c}_1 & \cdots & \vec{c}_5
\end{bmatrix} \vec{x} \approx \vec{b}
\]

We can treat the vector \(\vec{b} \in \mathbb{R}^N\) as a noisy measurement of the vector \(\vec{x}\), with measurement matrix \(M\) and some additional noise in it as well.

You also know that the true \(\vec{x}\) is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original \(\vec{x}\) as best we can.

However, your intern has managed to lose not only the measurements \(\vec{b}\) but the entire measurement matrix \(M\) as well!

Fortunately, you have found a backup in which you have all the pairwise inner products \(\langle \vec{c}_i, \vec{c}_j \rangle\) between the columns of \(M\) and each other as well as all the inner products \(\langle \vec{c}_i, \vec{b} \rangle\) between the columns of \(M\) and the vector \(\vec{b}\). Finally, you also know the inner product \(\langle \vec{b}, \vec{b} \rangle\) of \(\vec{b}\) with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

<table>
<thead>
<tr>
<th>(\langle \cdot, \cdot \rangle)</th>
<th>(\vec{c}_1)</th>
<th>(\vec{c}_2)</th>
<th>(\vec{c}_3)</th>
<th>(\vec{c}_4)</th>
<th>(\vec{c}_5)</th>
<th>(\vec{b})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\vec{c}_1)</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\vec{c}_2)</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>(\vec{c}_3)</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\vec{c}_4)</td>
<td>2</td>
<td>-1</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\vec{c}_5)</td>
<td>2</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\vec{b})</td>
<td>29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(So, for example, if you read this table, you will see that the inner product \(\langle \vec{c}_2, \vec{c}_3 \rangle = 1\), that the inner product \(\langle \vec{c}_3, \vec{b} \rangle = 2\), and that the inner product \(\langle \vec{b}, \vec{b} \rangle = 29\). By symmetry of the real inner product, \(\langle \vec{c}_3, \vec{c}_2 \rangle = 1\) as well.)

Your goal is to find which entries of \(\vec{x}\) are non-zero and what their values are.

(a) Use the information in the table above to answer which of the \(\vec{c}_1, \ldots, \vec{c}_5\) has the largest magnitude inner product with \(\vec{b}\)?

**Answer:**

Reading off the table, \([\vec{c}_4]\) has the largest inner product with \(\vec{b}\).

(b) Let the vector with the largest magnitude inner product with \(\vec{b}\) be \(\vec{c}_a\). Let \(\vec{b}_p\) be the projection of \(\vec{b}\) onto \(\vec{c}_a\). Write \(\vec{b}_p\) symbolically as an expression only involving \(\vec{c}_a, \vec{b}\), and their inner products with themselves and each other.

**Answer:**

The magnitude of the projection is \(\frac{\langle \vec{c}_a, \vec{b} \rangle}{\| \vec{c}_a \|}\), and the direction of the projection is \(\frac{\vec{c}_a}{\| \vec{c}_a \|}\). Thus:

\[
\vec{b}_p = \frac{\langle \vec{c}_a, \vec{b} \rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \vec{c}_a
\]
(c) Use the information in the table above to find which of the column vectors \( \vec{c}_1, \ldots, \vec{c}_5 \) has the largest magnitude inner product with the residue \( \vec{b} - \vec{b}_p \).

**Hint:** The linearity of inner products might prove useful.

**Answer:**

The inner product of \( \vec{b} - \vec{b}_p \) with a vector, say, \( \vec{c}_i \) is:

\[
\langle \vec{b} - \vec{b}_p, \vec{c}_i \rangle = \langle \vec{b}, \vec{c}_i \rangle - \frac{\langle \vec{c}_a, \vec{b} \rangle}{\langle \vec{c}_a, \vec{c}_a \rangle} \langle \vec{c}_a, \vec{c}_i \rangle
\]

Recall that \( \vec{c}_a \) to be the vector with the largest magnitude inner product with \( \vec{b} \). From part (a), we know this was \( \vec{c}_4 \). So let’s plug in \( \vec{c}_a = \vec{c}_4 \).

\[
= \langle \vec{b}, \vec{c}_1 \rangle - \frac{\langle \vec{c}_4, \vec{b} \rangle}{\langle \vec{c}_4, \vec{c}_4 \rangle} \langle \vec{c}_4, \vec{c}_i \rangle
\]

We can compute some of these terms from the table, to end up with

\[
= \langle \vec{b}, \vec{c}_1 \rangle - 3 \times \langle \vec{c}_4, \vec{c}_i \rangle
\]

Now all we need to do is find \( i \in \{1, 2, 3, 4, 5\} \) that maximizes the above expression.

Finding the numerical values of the inner products:

\[
\begin{align*}
\langle \vec{b}, \vec{c}_1 \rangle &= 4 \\
\langle \vec{b}, \vec{c}_2 \rangle &= -2 \\
\langle \vec{b}, \vec{c}_3 \rangle &= 2 \\
\langle \vec{b}, \vec{c}_4 \rangle &= 0 \\
\langle \vec{b}, \vec{c}_5 \rangle &= 2
\end{align*}
\]

Thus the vector with the highest inner product with the residue is: \( \vec{c}_1 \)

(d) Suppose that the vectors we found in parts (a) and (c) are \( \vec{c}_a \) and \( \vec{c}_c \). These correspond to the components of \( \vec{x} \) that are non-zero, that is, \( \vec{b} \approx x_a \vec{c}_a + x_c \vec{c}_c \). However, there might be noise in the measurements \( \vec{b} \), so we want to find the linear least squares estimates \( \hat{x}_a \) and \( \hat{x}_c \). Write a matrix expression for \( \begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix} \) in terms of appropriate matrices filled with the inner products of \( \vec{c}_a, \vec{c}_c, \vec{b} \).

**Answer:**

We use least squares to solve for \( \begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix} \). Let \( \mathbf{A} = \begin{bmatrix} \vec{c}_a & \vec{c}_c \end{bmatrix} \). Using the least-squares formula,

\[
\begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix} = (\mathbf{A}^\mathbf{T} \mathbf{A})^{-1} \mathbf{A}^\mathbf{T} \vec{b}
\]

\[
= \begin{bmatrix} \langle \vec{c}_a, \vec{c}_a \rangle & \langle \vec{c}_a, \vec{c}_c \rangle \\ \langle \vec{c}_c, \vec{c}_a \rangle & \langle \vec{c}_c, \vec{c}_c \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle \vec{c}_a, \vec{b} \rangle \\ \langle \vec{c}_c, \vec{b} \rangle \end{bmatrix}
\]

(e) Compute the numerical values of \( \hat{x}_a \) and \( \hat{x}_c \) using the information in the table.

**Answer:**

Substituting the previous expression with values from the table, we get: \( x_1 = 2 \frac{2}{3}, x_4 = 4 \frac{1}{3} \).
\[
\begin{bmatrix}
\hat{x}_4 \\
\hat{x}_1
\end{bmatrix} = \begin{bmatrix}
\langle \vec{c}_4, \vec{c}_4 \rangle & \langle \vec{c}_4, \vec{c}_1 \rangle \\
\langle \vec{c}_1, \vec{c}_4 \rangle & \langle \vec{c}_1, \vec{c}_1 \rangle
\end{bmatrix}^{-1} \begin{bmatrix}
\langle \vec{c}_4, \vec{b} \rangle \\
\langle \vec{c}_1, \vec{b} \rangle
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}^{-1} \begin{bmatrix}
6 \\
1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
6 \\
1
\end{bmatrix} = \frac{13}{3}
\]