1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:
- An \( n \times d \) dimension signature matrix \( S \) with columns \( \vec{S}_i \)
- An \( n \) dimension measurement vector \( \vec{y} \)
- The sparsity level \( m \) of the signal

Outputs:
- An estimate \( \hat{\vec{x}} \) in \( \mathbb{R}^d \) of the ideal message
- A set \( \Lambda_m \) containing \( m \) elements from \( \{1, 2, \ldots, d\} \)
- An \( n \)-dimensional approximation \( \hat{\vec{y}} \) of the measurement vector \( \vec{y} \)
- An \( n \)-dimensional residual \( \vec{r} = \vec{y} - \hat{\vec{y}} \)

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1: procedure OMP(S, \vec{y}, m)
2: \vec{r} \leftarrow \vec{y} \quad \text{(portion of measurement to be explained)}
3: A_0 \leftarrow [] \quad \text{(signatures found in the measurement)}
4: j \leftarrow 1 \quad \text{(iteration variable)}
5: \Lambda_0 \leftarrow [] \quad \text{(indices of vectors found)}
6: while \vec{r} \neq \vec{0} do
7:    procedure FIND SIGNATURE(\vec{r}, S)
8:        k \leftarrow \text{argmax}_i \langle \vec{S}_i, \vec{r} \rangle
9:        A_j \leftarrow [A_{j-1}, \vec{S}_k]
10:       \Lambda_j \leftarrow \Lambda_{j-1} \cup k
11:    end procedure
12:    procedure FIND PROJECTION(A_j, \vec{y})
13:        \hat{\vec{x}}_j = (A_j^T A_j)^{-1} A_j^T \vec{y}
14:        \hat{\vec{y}}_j = A_j \hat{\vec{x}}_j
15:        \vec{r} \leftarrow \vec{y} - \hat{\vec{y}}_j
16:    end procedure
17:    j \leftarrow j + 1
18: end while
19: The estimate for the ideal signal has non-zero indices at the components listed in \( \Lambda_m \). The value of \( \vec{x} \) in component \( \lambda_j \) equals the \( j \)th component of \( \hat{\vec{x}} \).
20: end procedure
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2. One Magical Procedure (Fall 2015 Final)

Suppose that we have a vector \( \vec{x} \in \mathbb{R}^5 \) and an \( N \times 5 \) measurement matrix \( M \) defined by column vectors
\(c_1, \ldots, c_5\), such that:

\[
M \bar{x} = \begin{bmatrix}
| & | \\
c_1 & \cdots & c_5 \\
| & |
\end{bmatrix} \bar{x} \approx \bar{b}
\]

We can treat the vector \(\bar{b} \in \mathbb{R}^N\) as a noisy measurement of the vector \(\bar{x}\), with measurement matrix \(M\) and some additional noise in it as well.

You also know that the true \(\bar{x}\) is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original \(\bar{x}\) as best we can.

However, your intern has managed to lose not only the measurements \(\bar{b}\) but the entire measurement matrix \(M\) as well!

Fortunately, you have found a backup in which you have all the pairwise inner products \(\langle c_i, c_j \rangle\) between the columns of \(M\) and each other as well as all the inner products \(\langle c_i, \bar{b} \rangle\) between the columns of \(M\) and the vector \(\bar{b}\). Finally, you also know the inner product \(\langle \bar{b}, \bar{b} \rangle\) of \(\bar{b}\) with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

<table>
<thead>
<tr>
<th>\langle \cdot, \cdot \rangle</th>
<th>\bar{c}_1</th>
<th>\bar{c}_2</th>
<th>\bar{c}_3</th>
<th>\bar{c}_4</th>
<th>\bar{c}_5</th>
<th>\bar{b}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\bar{c}_1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>\bar{c}_2</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>\bar{c}_3</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\bar{c}_4</td>
<td>2</td>
<td>-1</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\bar{c}_5</td>
<td>2</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\bar{b}</td>
<td>29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(So, for example, if you read this table, you will see that the inner product \(\langle \bar{c}_2, \bar{c}_3 \rangle = 1\), that the inner product \(\langle \bar{c}_3, \bar{b} \rangle = 2\), and that the inner product \(\langle \bar{b}, \bar{b} \rangle = 29\). By symmetry of the real inner product, \(\langle \bar{c}_3, \bar{c}_2 \rangle = 1\) as well.)

Your goal is to find which entries of \(\bar{x}\) are non-zero and what their values are.

(a) Use the information in the table above to answer which of the \(\bar{c}_1, \ldots, \bar{c}_5\) has the largest magnitude inner product with \(\bar{b}\)?

(b) Let the vector with the largest magnitude inner product with \(\bar{b}\) be \(\bar{c}_a\). Let \(\bar{b}_p\) be the projection of \(\bar{b}\) onto \(\bar{c}_a\). Write \(\bar{b}_p\) symbolically as an expression only involving \(\bar{c}_a, \bar{b}\), and their inner products with themselves and each other.

(c) Use the information in the table above to find which of the column vectors \(\bar{c}_1, \ldots, \bar{c}_5\) has the largest magnitude inner product with the residue \(\bar{b} - \bar{b}_p\).

*Hint:* The linearity of inner products might prove useful.

(d) Suppose that the vectors we found in parts (a) and (c) are \(\bar{c}_a\) and \(\bar{c}_c\). These correspond to the components of \(\bar{x}\) that are non-zero, that is, \(\bar{b} \approx x_a \bar{c}_a + x_c \bar{c}_c\). However, there might be noise in the measurements \(\bar{b}\), so we want to find the linear least squares estimates \(\hat{x}_a\) and \(\hat{x}_c\). Write a matrix expression for \(\begin{bmatrix} \hat{x}_a \\ \hat{x}_c \end{bmatrix}\) in terms of appropriate matrices filled with the inner products of \(\bar{c}_a, \bar{c}_c, \bar{b}\).

(e) Compute the numerical values of \(\hat{x}_a\) and \(\hat{x}_c\) using the information in the table.