EECS 16A Designing Information Devices and Systems I
Spring 2017 Official Lecture Notes **Note 18**

Code Division Multiple Access

In many real world scenarios, measuring an isolated variable or signal is infeasible. Consider the case of CDMA (Code Division Multiple Access). In cellular networks, the geographical area is divided into cells which are served by a single cell tower. Users in a particular cell talk to the tower associated with that cell in order to communicate with other users (placing a call, texting) or to fetch content from the internet. Users use various techniques in order to avoid interfering with other users in the cell. (There is also the matter of interference with users from other cells but we will not talk about it in this notes.) One such technique used to separate users is CDMA. CDMA is a "Spread-spectrum" technology in which each user is allocated a special code which serves as its "signature". The *message signal* to be transmitted changes at a much slower rate when compared to that of the *signature code*. The *message signal* is can be any real number and the *signature codes* are streams of ± 1. The *signature codes* have a high auto-correlation values and the cross-correlation values with other codes are very low. The *signature codes* are known by each user as well as the tower. The *signature codes* are multiplied with the appropriate *message signal* to get the transmission signal. The signal received at the tower is a linear combination of signals from all transmitting users. We assume that most of the times, the number of users transmitting at the same time is a small fraction of the total number of users in the cell.

Let T_m be the "message time" which is the time for each message and T_c be the "code time" which is the time for each new symbol of the code to be generated. In the figure $T_m = 14T_c$. In this note, the relationship between T_m and T_c is given by $T_m = k \times T_c$ and k is called the signature length.

Let the *signature code* for user i in a cell be represented by ϕ_i. Consider a single "message time". Let the message transmitted in this single "message time" by all the users in the cell be given by the vector $\vec{x}$ where $\vec{x}[i]$ denotes the message transmitted by user i. Let $\vec{b}$ denote the signal received at the tower in this message time. Then we have the following relationship:

$$\vec{b} = \Phi \vec{x} \tag{1}$$

where Φ is the matrix whose i^{th} column is a circularly shifted version of ϕ_i. How can one go about using $\vec{b}$ to extract the signal $\vec{x}$?

As a first attempt we could correlate the received signal $\vec{b}$ with each of the signatures that could possibly be and take the peak correlation value to be the message value. Let us look at this example in which there are about 2000 users in the cell and only 3 users (user 40, 100 and 312) transmit and their corresponding messages are 10, 10 and 8. The result of the correlation is shown in Fig. **??**.

We see that although we do see the peaks for all the users which are present, the values aren't quite right. Let us look at another example in which 4 users (user 40, 100, 312 and 350) transmit and their corresponding messages are 100, 10, 8 and 0.02. The result of the correlation with different signatures is shown in Fig. **??**.

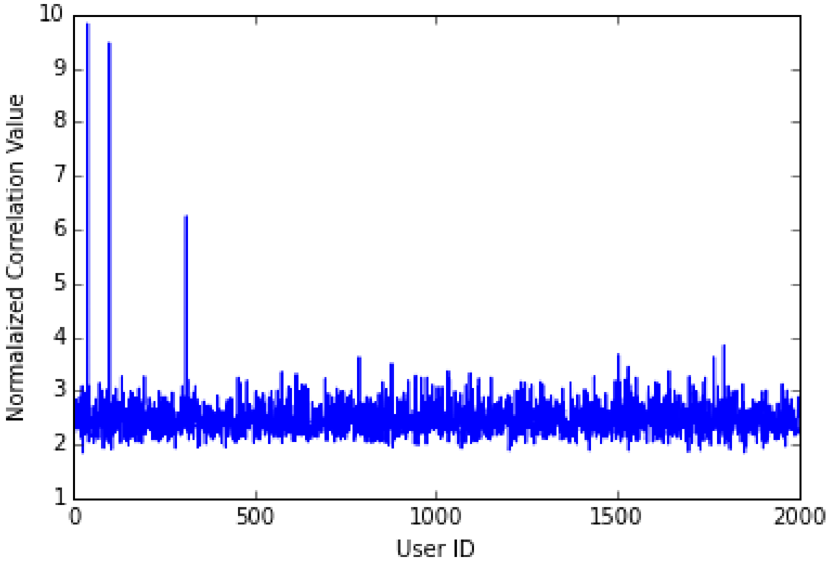


Figure 1: Maximum correlation of received signal with circular shifts of all the signatures

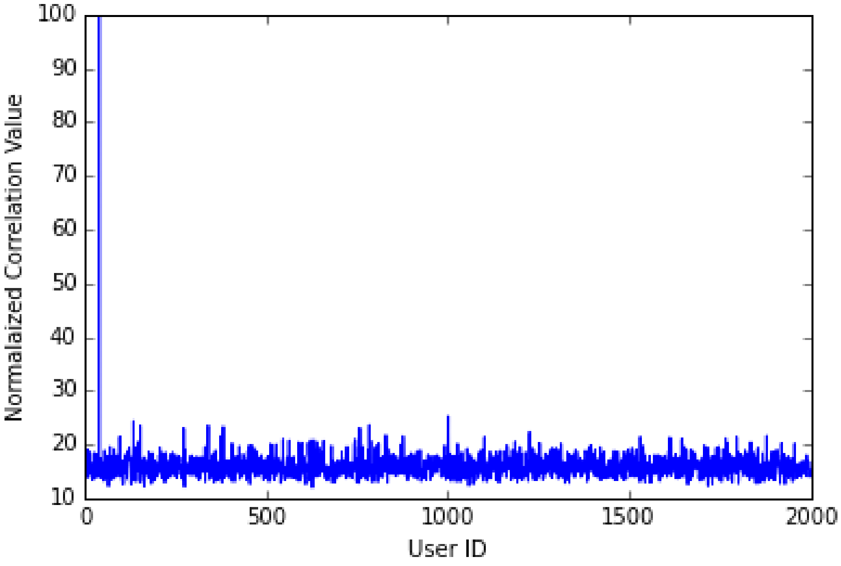


Figure 2: Maximum correlation of received signal with circular shifts of all the signatures

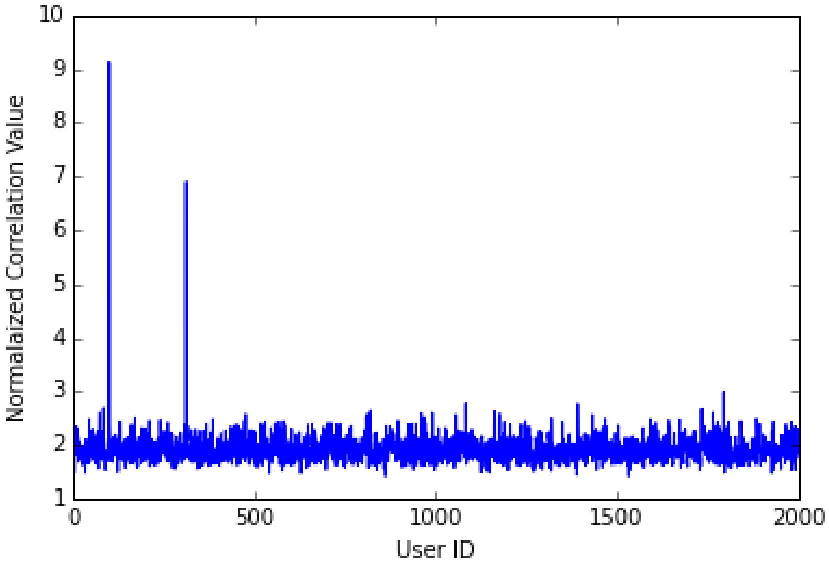


Figure 3: Maximum correlation of residue of the received signal after genie aided removal with circular shifts of all the signatures

The peaks corresponding to users 100, 312 and 350 don't seem to appear at all! What could possibly cause this? Perhaps the very haigh value of user 40's signal cross-correlated enough with other signatures to kill the peaks corresponding to users 100, 312 and 350. What if a genie told us the level of the highest signal (user 40) is 100. Maybe killing the signal will result in canceling of this interference and we could see the other peaks. After removing the signal corresponding to user 40, the result of the correlation with the signatures is shown in Fig. **??**

We now see the peaks correspnonding to users 100 and 312. But we do not usually have a genie to tell us the signal level. Then what could serve as a good proxy for the actual signal level? Let us try the max correlation value which is the projection of the recieved signal onto that signature space. The results of correlations of the received signal in the first step and residues in the subsequent steps is shown in Fig. **??**

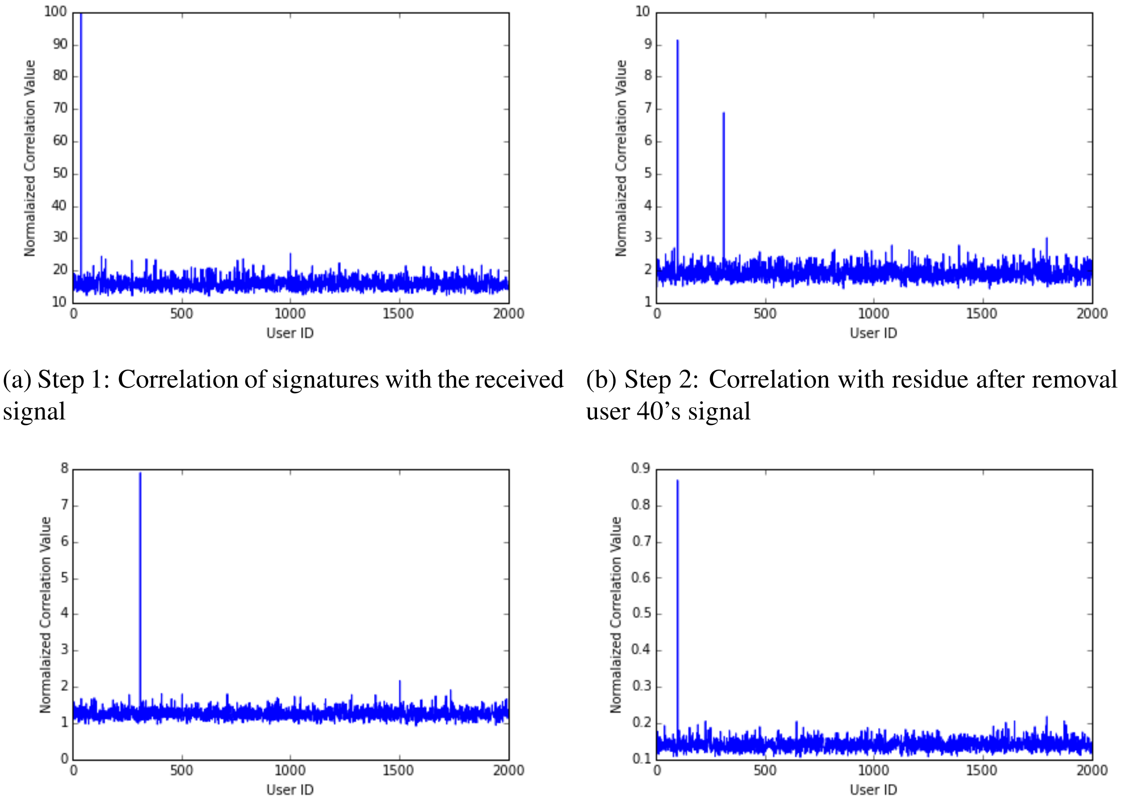
In Fig. **??** we would have expected to find a peak at user 350 but we surprisingly we don't. Why is that the case? When can we not find the peak where it should have been? The reason was that we were removing correlated signal from the measurement as if they were orthogonal which is not the case here. The right approach would be to remove the *projection* of the received signal onto the subspace spanned by the signatures (the least-squares approach would give us the projection of the received signal onto a subspace).

If we project the received signal onto the subspace spanned by the signatures of user 40, 100 and 312 and remove it from the received signal and look for a unique signature again, we get Fig. **??**.

Orthogonal Matching Pursuit

The discussion in the previous section gives us an algorithm for extracting information from sparse signal. This algorithm is called Orthogonal Matching Pursuit (OMP).

Suppose $\vec{x} \in \mathbb{R}^d$ is the sparse vector we want to measure, with m non-zero entries (its sparsity). Let $\mathbf{S}$ be the



(a) Step 1: Correlation of signatures with the received signal

(b) Step 2: Correlation with residue after removal of user 40's signal

(c) Step 3: Correlation with residue after removal of user 100's signal

(d) Step 4: Correlation with residue after removal of user 312's signal

Figure 4: Maximum correlation of received signal with circular shifts of all the signatures

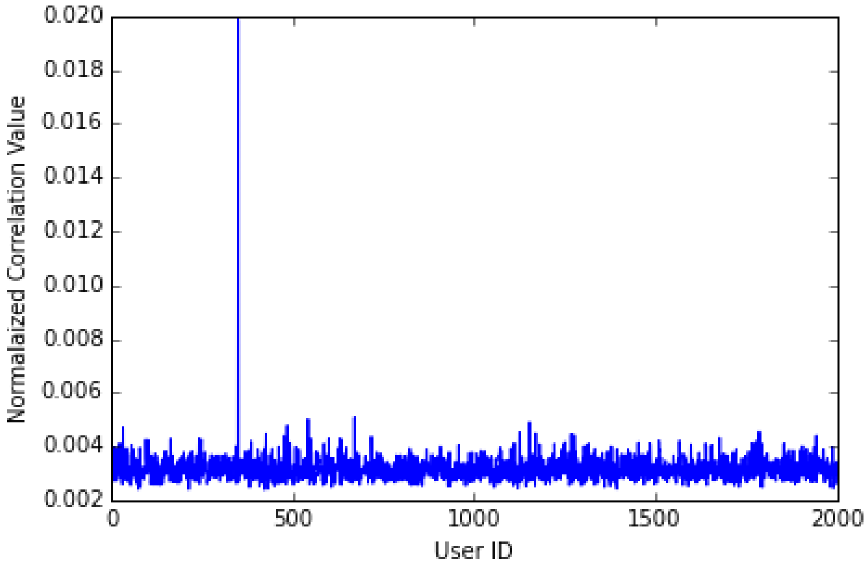


Figure 5: Maximum correlation of residue of the received signal after removal of the projection of the received signal onto the subspace spanned by the signatures found so far

$n \times d$ measurement matrix with columns $\vec{S}_i$ such that:

$$\begin{bmatrix} \vec{S}_1 & \cdots & \vec{S}_d \end{bmatrix} \vec{x} = \vec{y}$$

where the vector $\vec{y} \in \mathbb{R}^n$ is the results from the n measurements of $\vec{x}$. We can recover the signal exactly if there are exactly d (linearly independent) measurements ($n = d$), but OMP works even when the number of measurements is less than the number of dimensions ($n < d$).

Here is a high-level description of OMP:

1. Find the vector $\vec{S}_*$ (not chosen before) with the highest correlation with $\vec{y}$.
2. Use least squares to project $\vec{y}$ onto the subspace spanned by the vectors already found in previous steps and $\vec{S}_*$, then find the residue $\vec{r}$ (this is the quantity minimized by least squares).
3. Repeat the above steps using $\vec{r}$ in place of $\vec{y}$ for a total of m times, the sparsity of vector $\vec{x}$.

First, what does it mean for a vector $\vec{S}_*$ to have the highest correlation with $\vec{y}$? Geometrically, this means that the angle between $\vec{S}_*$ and $\vec{y}$ is the smallest amongst all vectors $\vec{S}_i$. This translates to $\underset{i}{max} \frac{\langle \vec{S}_i, \vec{y} \rangle}{||\vec{S}_i|| \cdot ||\vec{y}||}$, which is equivalent to $\underset{i}{max} \left\langle \vec{S}_i, \vec{y} \right\rangle$ when we assume all vectors $\vec{S}_i$ are normalized. Next, to project $\vec{y}$ onto the subspace spanned by the selected vectors, we create a matrix A with the selected vectors as its columns, and find the least squares estimate $\hat{x}$ minimizing $||A\hat{x} - \vec{b}||^2$. For example, if the selected vectors are $\vec{S}_2$ and $\vec{S}_4$, the minimized quantity is $\left|\left| \begin{bmatrix} \vec{S}_2 & \vec{S}_4 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_4 \end{bmatrix} - \vec{b} \right|\right|^2$. The following section precisely describes the implementation of OMP.

Setup:
Let the number of possible unique signatures (and users) be d and the unique signatures be represented by $\vec{S}_i$ (of length n). Each signature can potentially carry a message α_i along with it. Then the measurement at the receiver is

$$\vec{y} = \sum_{i=1}^{d} \alpha_i \vec{S}_i \tag{2}$$

We will assume that the number of users talking at the same time is very small *i.e.,* there are at most m non-zero α_is. The OMP algorithm is described below.

1: **Inputs:**

- An $n \times d$ dimension signature matrix $\mathbf{S}$ with columns $\vec{S}_i$
- An n dimension measurement vector $\vec{y}$
- The sparsity level m of the signal

2: **Outputs:**

- An estimate $\hat{\alpha}$ in $\mathbb{R}^d$ of the ideal message
- A set Λ_m containing m elements from $\{1, 2, \ldots, d\}$
- An n-dimensional approximation $\vec{y}_{Am}$ of the measurement vector $\vec{y}$
- An n-dimensional residual $\vec{r} = \vec{y} - \vec{y}_{\mathbf{A}_m}$

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procedure OMP($\mathbf{S}$, $\vec{y}$, $m$)
    $\vec{r} \leftarrow \vec{y}$ (portion of measurement to be explained)
    $\mathbf{A}_0 \leftarrow [\,]$ (signatures found in the measurement)
    $j \leftarrow 1$ (interation variable)
    $\Lambda_0 \leftarrow [\,]$ (vectors found)
    while $\vec{r} \neq \vec{0}$ do
        procedure FIND SIGNATURE($\vec{r}$, $\mathbf{S}$)
            $k \leftarrow \underset{i}{\operatorname{argmax}} \langle \vec{S}_i, \vec{r} \rangle$
            $\vec{v}_j \leftarrow \vec{S}_k$
            $\mathbf{A}_j \leftarrow [\mathbf{A_{j-1}} \vec{v}_j]$
            $\Lambda_j \leftarrow \Lambda_{j-1} \cup k$
        end procedure
        procedure FIND PROJECTION($\mathbf{A}_j$, $\vec{y}$)
            $\vec{y}_{\mathbf{A}_j} \leftarrow \mathbf{A}_j (\mathbf{A}_j^T \mathbf{A}_j)^{-1} \mathbf{A}_j^T \vec{y}$
            $\vec{r} \leftarrow \vec{y} - \vec{y}_{\mathbf{A}_j}$
        end procedure
        $j \leftarrow j + 1$
    end while
    $\hat{x} \leftarrow (\mathbf{A}_m^T \mathbf{A}_m)^{-1} \mathbf{A}_m^T \vec{y}$
    The estimate for the ideal signal has nonzero indices at the components listed in $\Lambda_m$. The value of
the estimate $\hat{\alpha}$ in component equals $\lambda_j$ the $j$th component of $\hat{x}$
end procedure
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