EE16B MF Section - Week 2

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Warmup

Use the matrix form of the DFT to find the DFT of the function:

$$x(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

Use $N=3$. Then use the inverse DFT on the result, again using the matrix form.

Solution:

DFT

$$\underline{X} = F\underline{x} \qquad \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-i\omega_0} & e^{-i(2\omega_0)} \\ 1 & e^{-i(2\omega_0)} & e^{-i(4\omega_0)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

IDFT:

$$\underline{x} = \frac{1}{N} F^* \underline{X}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\omega_0} & e^{2i\omega_0} \\ 1 & e^{2i\omega_0} & e^{4i\omega_0} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1+1+1 \\ 1+e^{i2\pi/3}+e^{i4\pi/3} \\ 1+e^{i4\pi/3}+e^{i8\pi/3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

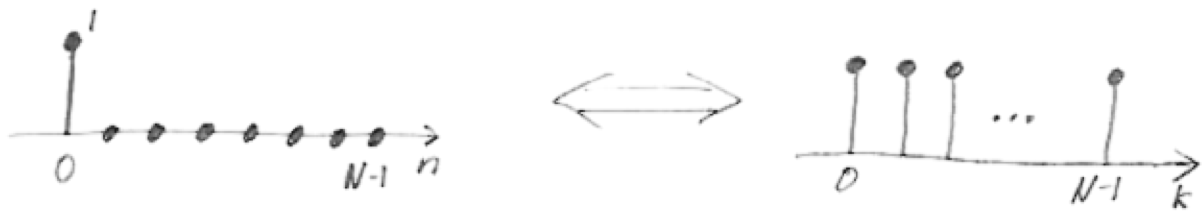
Hooray, it worked!

iPython interlude: fft, rfft, ifft, oh my!

Exploring the time-frequency duality

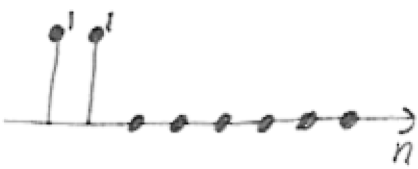
* In general, short spikes in the time domain correspond to broad frequency spectra, and vice versa!

Recall our "unit impulse" example:



What if we add a second impulse?

Ex. 2



Let's get to it: $X_k = \sum_{n=0}^{N-1} x(n) e^{-ik\omega_0 n}$

Take $N=8 \Rightarrow \omega_0 = \pi/4$

$k=0 \quad X_0 = \underset{n=0}{e^{-i0\omega_0 0}} + \underset{n=1}{e^{-i0\omega_0 1}} = 1+1 = 2$

$k=1 \quad X_1 = e^{-i1\omega_0 0} + e^{-i1\omega_0 1}$

$= 1 + e^{-i\pi/4}$

$= 1 + \cos(-\pi/4) + i\sin(-\pi/4)$

$= 1 + \sqrt{2}/2 + (-\sqrt{2}/2\, i)$

Hm, it's complex... how about:

$|X_1| \approx 1.8$

$$X_2 = e^{-i2\omega_0 0} + e^{-i2\omega_0 1}$$
$$= 1 + e^{-i\pi/2}$$
$$= 1 + i$$
$$|X_2| \approx 1.4$$

$$X_3 = 1 + e^{-i3\omega_0}$$
$$= 1 + e^{-i3\pi/4}$$
$$= 1 - \sqrt{2}/2 - \sqrt{2}/2\, i$$
$$|X_3| \approx 0.4$$

$$X_4 = 1 + e^{-i\pi} = 0$$

$$X_5 = 1 + e^{-i5\omega_0}$$
$$= 1 + e^{-i5\pi/4}$$
$$= 1 - \sqrt{2}/2 + \sqrt{2}/2\, i$$
$$|X_5| \approx 0.4$$

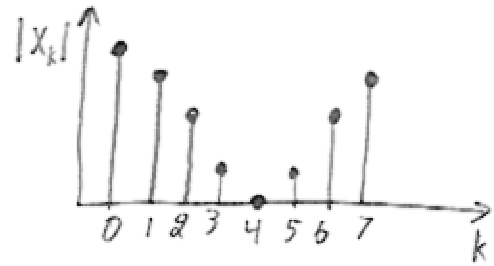
Hm, this looks familiar... $X_5 = X_3^*$, and $|X_5| = |X_3|$!
This is because the time-domain signal is real-valued, so its spectrum is (nearly) symmetric - each coefficient X_k has a partner, related by complex conjugation.

So actually, we can save some time:

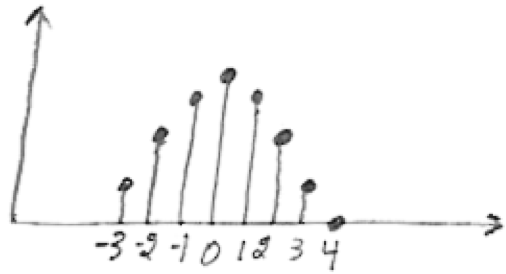
$$X_6 = X_2^* = 1 - i$$
$$|X_6| \approx 1.4$$

$$X_7 = X_1^* = 1 + \sqrt{2}/2 + \sqrt{2}/2\, i$$
$$|X_7| \approx 1.8$$

Then the spectrum plot of the magnitudes looks like:



or ...



Ex 3

Find the width-8 DFT of this signal:

$$X_k = \sum_{n=0}^{N-1} x(n) e^{-ik\omega_0 n}$$

$$\omega_0 = \pi/4$$

$$X_0 = 3$$

$$X_1 = 1 + e^{-i\omega_0} + e^{-2i\omega_0}$$
$$= 1 + e^{-i\pi/4} + e^{-i\pi/2}$$
$$= 1 + \left(\sqrt{2}/2 - \sqrt{2}/2\, i\right) - i$$
$$= \left(1 + \sqrt{2}/2\right) + \left(1 + \sqrt{2}/2\right) i$$
$$|X_1| \approx 2.4$$

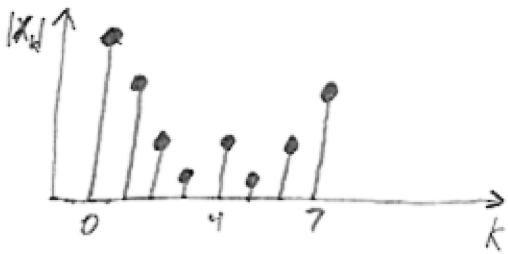
$$X_2 = 1 + e^{-2i\omega_0} + e^{-4i\omega_0}$$
$$= 1 + e^{-i\pi/2} + e^{-i\pi}$$
$$= 1 - i - 1 = -i$$
$$|X_2| = 1$$

$$X_3 = 1 + e^{-i3\omega_0} + e^{-i6\omega_0}$$
$$= 1 + e^{-i3\pi/4} + e^{-i3\pi/2}$$
$$= 1 - \sqrt{2}/2 - \sqrt{2}/2\, i + i$$
$$= \left(1 - \sqrt{2}/2\right) + \left(1 - \sqrt{2}/2\right) i$$
$$|X_3| \approx .41$$

$$X_4 = 1 + e^{-i4\omega_0} + e^{-i8\omega_0}$$
$$= 1 + e^{-i\pi} + e^{-i2\pi}$$
$$= 1 - 1 + 1 = 1$$

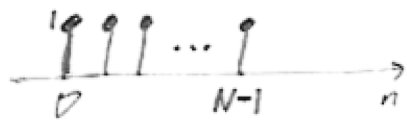
$$X_5 = X_3^*, \quad X_6 = X_2^*, \quad X_7 = X_1^*$$

So the spectrum looks like:

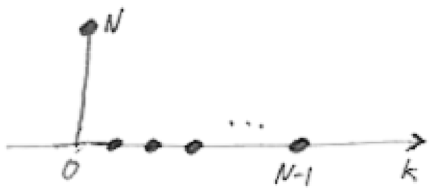


From these examples, we can see that as we add additional "lollipops" in time, the frequency spectrum becomes more and more clustered around 0.

What if we extend this process all the way? What is the DFT of:



Answer:



Why does this make sense? Why is $X_0 = N$ and not 1?

iPython interlude #2: The stock market, revisited

The bottom line:

There are many "dualities" between the time and frequency representations of signals! "The same" or "opposite" are usually pretty good guesses about the relationships between them.