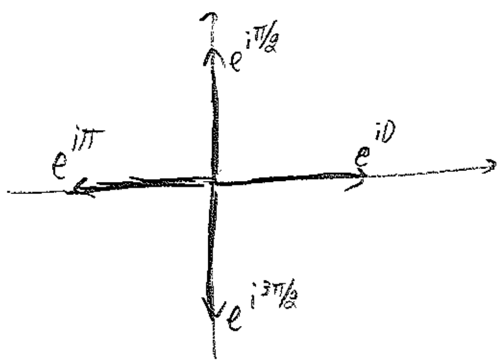
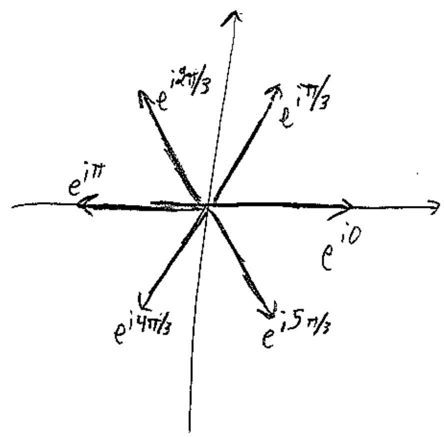
EE16B Section 2B: Variance and Covariance

<u>Warmup</u>: We can think of each DFT basis function $\Psi_k = \frac{1}{N} e^{ik\omega_0 n}$ as being associated with a particular frequency $e^{ik\omega_0}$. Label and plot the frequency associated with each Ψ_k as a phasor in the complex plane for:

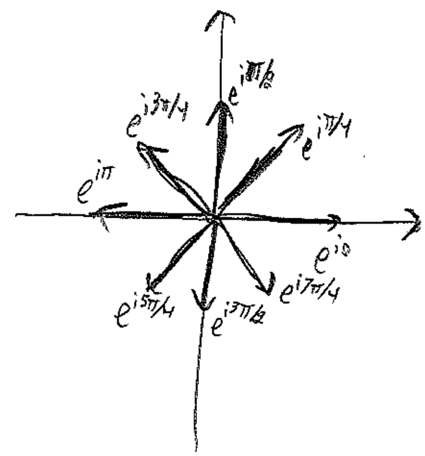
$N=4$



$N=6$



$N=8$



Questions from Lecture

iPython Interlude

Variance and Covariance

If we have some data set $A = \{a_0, a_1, \ldots, a_{N-1}\}$ that has a mean of zero, then its <u>variance</u> can be defined as

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} a_i^2$$

If we have a second zero-mean data set $B = \{b_0, b_1, \ldots, b_{N-1}\}$, then we can define their <u>covariance</u> as

$$\sigma_{AB}^2 = \frac{1}{N} \sum_{i=0}^{N-1} a_i b_i$$

<u>Example</u>: (Experiment in section: Measure heights and armspans, and find σ_x^2, σ_y^2, and σ_{xy}^2.)

A brief aside: We can define the <u>correlation coefficient</u> of two data sets X and Y as

$$r_{xy} = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y} = \frac{\sigma_{xy}^2}{\sqrt{\sigma_x^2}\sqrt{\sigma_y^2}}$$

<u>Example cont'd</u>: (Find r^2 for the data in section.)

Let's now move into matrix-vector land. We'll redefine X and y as vectors $\underline{x}$ and $\underline{y}$. Then we can find the covariance by:

$$\sigma_{xy}^2 = \frac{1}{n-1} \underline{x}^T \underline{y}$$

(For obsure reasons, we will use $1/n-1$ instead of $1/n$ to normalize from here on.)

So for two data sets, there are only 3 possible covariance operations: σ_{xx}^2, σ_{yy}^2, and σ_{xy}^2. What if you had 3 data sets and you wanted to find all possible combinations? Starts getting tedious... let's move to matrices for bookkeeping.

(assume $\underline{x}, \underline{y}, \underline{z}$ are zero-mean column vectors of length n):

$$X = \begin{bmatrix} \underline{x}^T \\ \underline{y}^T \\ \underline{z}^T \end{bmatrix} \qquad X^T = \begin{bmatrix} \underline{x} & \underline{y} & \underline{z} \end{bmatrix}$$

Multiply them to find all the covariances!

$$S_x = \frac{1}{n-1} X X^T = \begin{bmatrix} \underline{x}^T \\ \underline{y}^T \\ \underline{z}^T \end{bmatrix} \begin{bmatrix} \underline{x} & \underline{y} & \underline{z} \end{bmatrix} = \begin{bmatrix} \underline{x}^T\underline{x} & \underline{x}^T\underline{y} & \underline{x}^T\underline{z} \\ \underline{y}^T\underline{x} & \underline{y}^T\underline{y} & \underline{y}^T\underline{z} \\ \underline{z}^T\underline{x} & \underline{z}^T\underline{y} & \underline{z}^T\underline{z} \end{bmatrix}$$

But based on our previous definition, this means that

$$S_X = \begin{bmatrix} \sigma^2_x & \sigma^2_{xy} & \sigma^2_{xz} \\ \sigma^2_{xy} & \sigma^2_y & \sigma^2_{yz} \\ \sigma^2_{xz} & \sigma^2_{yz} & \sigma^2_z \end{bmatrix}$$

This matrix is symmetric because $\sigma^2_{xy} = \sigma^2_{yx}$ - it represents all possible covariances of the three vectors. What is on the diagonal?

Addendum: Data for covariance calculations:

Height (cm): $h = \{161, 142, 148, 149\}$

Armspan (cm): $s = \{162, 139, 151, 148\}$

Zero-mean:

$$a = h - \bar{h} = \{11, -8, -2, -1\}$$

$$b = s - \bar{s} = \{12, -11, 1, -2\}$$

$$\sigma_a^2 = \frac{1}{4}\left(11^2 + (-8)^2 + (-2)^2 + (-1)^2\right) = 47.5$$

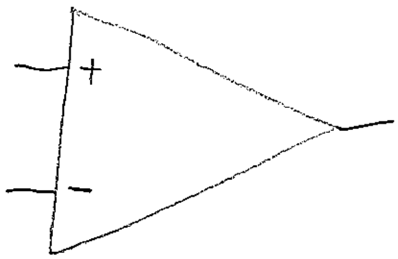
$$\sigma_b^2 = \frac{1}{4}\left(12^2 + (-11)^2 + (1)^2 + (-2)^2\right) = 67.5$$

$$\sigma_{ab}^2 = \frac{1}{4}\left(11 \cdot 12 + (-8)(-11) + (-2) \cdot 1 + (-1)(-2)\right) = 55$$

$$r = \frac{\sigma_{ab}^2}{\sigma_a \sigma_b} = \frac{55}{\sqrt{47.5}\sqrt{67.5}} = .918$$

Pretty well correlated!

Op-amps!



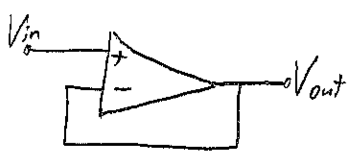
"Magic" circuit follows two golden rules:

1) Voltage difference between the two input terminals is 0.

2) Current into either input terminal is 0.

... as long as it is connected in <u>negative feedback</u>, with the − terminal (somehow) wired to the output!

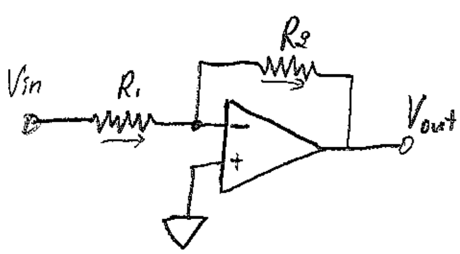
<u>Ex 1:</u>



$V_{out} = V_{in}$!

"voltage follower"

<u>Ex 2:</u>



"inverting amplifier"

Rule 1: $V^- = V^+ = 0$

Rule 2: $i_{R1} = i_{R2}$ (KCL)

$$\frac{V^- - V_{in}}{R_1} = \frac{V_{out} - V^-}{R_2}$$

$$\frac{-V_{in}}{R_1} = \frac{V_{out}}{R_2} \Rightarrow \frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}$$