Stability of Linear State Models Continued

The Scalar Case

In the last lecture we saw that the solution of the scalar equation

\[ x(t + 1) = ax(t) + bu(t) \]  \hspace{1cm} (1)

is:

\[ x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \quad t = 1, 2, 3, \ldots \]  \hspace{1cm} (2)

The first term \( a^t x(0) \) represents the effect of the initial condition and the second term \( \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \) represents the effect of the input sequence \( u(0), u(1), \ldots, u(t-1) \).

**Definition.** We say that a system is stable if its state \( x(t) \) remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is unstable if we can find an initial condition and a bounded input sequence such that \( |x(t)| \to \infty \) as \( t \to \infty \).

It follows from (2) that, if \( |a| > 1 \), then a nonzero initial condition \( x(0) \neq 0 \) in enough to drive \( |x(t)| \) unbounded. This is because \( |a|^t \) grows unbounded and, with \( u(t) = 0 \) for all \( t \), we get \( |x(t)| = |a^t x(0)| \to |a|^t |x(0)| \to \infty \). Thus, (1) is unstable for \( |a| > 1 \).

Next, we show that (1) is stable when \( |a| < 1 \) is stable. In this case \( a^t x(0) \) decays to zero, so we need only to show that the second term in (2) remains bounded for any bounded input sequence. A bounded input means we can find a constant \( M \) such that \( |u(t)| \leq M \) for all \( t \).

Thus,

\[ \left| \sum_{k=0}^{t-1} a^{t-1-k} bu(k) \right| \leq \sum_{k=0}^{t-1} |a|^{t-1-k} |b||u(k)| \leq |b|M \sum_{k=0}^{t-1} |a|^{t-1-k}. \]

Defining the new index \( s = t - 1 - k \) we rewrite the last expression as

\[ |b|M \sum_{s=0}^{t-1} |a|^s, \]

and note that \( \sum_{s=0}^{t-1} |a|^s \) is a geometric series that converges to \( \frac{1}{1-|a|} \) since \( |a| < 1 \). Therefore, each term in (2) is bounded and we conclude stability for \( |a| < 1 \).
Summary: The scalar system (1) is stable when $|a| < 1$, and unstable when $|a| > 1$.

When $a$ is a complex number, a perusal of the stability and instability arguments above show that the same conclusions hold if we interpret $|a|$ as the modulus of $a$, that is:

$$|a| = \sqrt{\text{Re}\{a\}^2 + \text{Im}\{a\}^2}.$$

What happens when $|a| = 1$? If we disallow inputs ($b = 0$), this case is referred to as “marginal stability” because $|a^t x(0)| = |x(0)|$, which neither grows nor decays. If we allow inputs ($b \neq 0$), however, we can find a bounded input to drive the second term in (2) unbounded. For example, when $a = 1$, the constant input $u(t) = 1$ yields:

$$\sum_{k=0}^{t-1} a^{t-1-k} bu(k) = \sum_{k=0}^{t-1} b = bt$$

which grows unbounded as $t \to \infty$. Therefore, $|a| = 1$ is a precarious case that must be avoided in designing systems.

**The Vector Case**

When $\vec{x}(t)$ is an $n$-dimensional vector governed by

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t),$$

recursive calculations lead to the expression

$$\vec{x}(t) = A^t \vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} Bu(k) \quad t = 1, 2, 3, \ldots$$

This is similar to (2), except that the scalars $a$ and $b$ are now replaced with matrices $A$ and $B$, and $a^t$ is replaced with the matrix power $A^t = A \cdot \cdots \cdot A$, $t$ times.

Unlike the scalar case (2), stability properties are not apparent from (4). However, when $A$ is diagonalizable we can employ the change of variables $\vec{z} \triangleq T \vec{x}$ and select the matrix $T$ such that

$$A_{\text{new}} = T A T^{-1}$$

is diagonal. $A$ and $A_{\text{new}}$ have the same eigenvalues and, since $A_{\text{new}}$ is diagonal, the eigenvalues appear as its diagonal entries:

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$
The state model for the new variables is

\[ Z(t + 1) = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} Z(t) + B_{\text{new}} u(t) \]  

(5)

which nicely decouples into scalar equations:

\[ z_i(t + 1) = \lambda_i z_i(t) + b_i u(t), \quad i = 1, \ldots, n \]  

(6)

where we denote by \( b_i \) the \( i \)-th entry of \( B_{\text{new}} \). Then, the results of the previous section imply stability when \(|\lambda_i| < 1\) and instability when \(|\lambda_i| > 1\).

For the whole system to be stable each subsystem must be stable, therefore we need \(|\lambda_i| < 1\) for each \( i = 1, \ldots, n \) for stability. If there exists at least one eigenvalue \( \lambda_i \) with \(|\lambda_i| > 1\) then we conclude instability because we can drive the corresponding state \( z_i(t) \) unbounded.

Summary: The discrete-time system (3) is stable if \(|\lambda_i| < 1\) for each eigenvalue \( \lambda_1, \ldots, \lambda_n \) of \( A \), and unstable if \(|\lambda_i| > 1\) for some eigenvalue \( \lambda_i \).

Although we assumed diagonalizability of \( A \) above, the same stability and instability conditions hold when \( A \) is not diagonalizable. In that case a transformation exists that brings \( A_{\text{new}} \) to an upper-diagonal form with eigenvalues on the diagonal\(^2\). Thus, instead of (5) we have

\[ Z(t + 1) = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & * \\ & & & \lambda_n \end{bmatrix} Z(t) + B_{\text{new}} u(t) \]  

(7)

where the entries marked with ‘*’ may be nonzero, but we don’t need their explicit values for the argument that follows. Then it is not difficult to see that \( z_n \) obeys

\[ z_n(t + 1) = \lambda_n z_n(t) + b_n u(t) \]  

(8)

which does not depend on other states, so we conclude \( z_n(t) \) remains bounded for bounded inputs when \(|\lambda_n| < 1\). The equation for \( z_{n-1} \) has the form

\[ z_{n-1}(t + 1) = \lambda_{n-1} z_{n-1}(t) + [ * z_n(t) + b_{n-1} u(t) ] \]  

(9)

where we can treat the last two terms in brackets as a bounded input since we have already shown that \( z_n(t) \) is bounded. If \(|\lambda_{n-1}| < 1\) we

\(^2\) The details of this transformation are beyond the scope of this course.
conclude $z_{n-1}(t)$ is itself bounded and proceed to the equation:

$$z_{n-2}(t + 1) = \lambda_{n-2} z_{n-2}(t) + [\star z_{n-1}(t) + \star z_n(t) + b_{n-2} u(t)]. \quad (10)$$

Continuing this argument recursively we conclude stability when $|\lambda_i| < 1$ for each eigenvalue $\lambda_i$.

To conclude instability when $|\lambda_i| > 1$ for some eigenvalue, note that the ordering of the eigenvalues in (7) is arbitrary: we can put them in any order we want by properly selecting $T$. Therefore, we can assume without loss of generality that an eigenvalue with $|\lambda_i| > 1$ appears in the $n$th diagonal entry, that is $|\lambda_n| > 1$. Then, instability follows from the scalar equation (8).

**Stability of Linear Continuous-Time State Models**

We will state analogous results for the continuous-time case without detailed derivations. The first-order differential equation

$$\frac{dx(t)}{dt} = ax(t) + bu(t)$$

admits the solution

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-s)} bu(s) ds,$$

which is analogous to (2). It is stable if $a < 0$ and unstable if $a > 0$. If we allow $a$ to be complex, these conditions become $\text{Re}\{a\} < 0$ and $\text{Re}\{a\} > 0$, respectively.

For the vector case

$$\frac{d}{dt} \bar{x}(t) = A \bar{x}(t) + B \bar{u}(t) \quad (11)$$

we use a change of variables that brings $A$ to a diagonal form when it is diagonalizable, and to an upper-diagonal form otherwise. Then arguments similar to the discrete-time case lead to the following conclusion:

**Summary:** The continuous-time system (11) is stable if $\text{Re}\{\lambda_i\} < 0$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of $A$, and unstable if $\text{Re}\{\lambda_i\} > 0$ for some eigenvalue $\lambda_i$. 