Stability of Linear State Models Continued

We have seen that the discrete time system
\[ x(t+1) = Ax(t) + Bu(t) \]  \hspace{1cm} (1)

is stable if \(|\lambda_i| < 1\) for each eigenvalue \(\lambda_1, \ldots, \lambda_n\) of \(A\), and unstable if \(|\lambda_i| > 1\) for some eigenvalue \(\lambda_i\).

The continuous time system
\[ \frac{d}{dt} \bar{x}(t) = A\bar{x}(t) + B\bar{u}(t) \]  \hspace{1cm} (2)

is stable if \(\text{Re}\{\lambda_i\} < 0\) for each eigenvalue \(\lambda_1, \ldots, \lambda_n\) of \(A\), and unstable if \(\text{Re}\{\lambda_i\} > 0\) for some eigenvalue \(\lambda_i\).

The figures below highlight the regions of the complex plane where the eigenvalues must lie for stability of a discrete-time (left) and continuous-time (right) system.

Example: In Lecture 4B we derived continuous-time linearized models for the downward and upright positions of the pendulum, and obtained:

\[ A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{\tau} & 1 \end{bmatrix} \quad A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{\tau} & \frac{-\ell}{m} \end{bmatrix}. \]  \hspace{1cm} (3)

The eigenvalues of \(A_{\text{down}}\) are the roots of \(\lambda^2 + \frac{k}{m} \lambda + \frac{\ell}{\tau}\), which can be shown to have strictly negative real parts when \(k > 0\). Thus the downward position is stable.
The eigenvalues of $A_{up}$ are the roots of $\lambda^2 + \frac{k}{m}\lambda - \frac{g}{\ell}$, which are given by:

$$\lambda_1 = -\frac{k}{2m} - \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{\ell}}, \quad \lambda_2 = -\frac{k}{2m} + \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{\ell}}.$$ 

Since $\lambda_2 > 0$, the upright position is unstable. Note that making the length $\ell$ smaller increases the value of $\lambda_2$. This suggests that a smaller length aggravates the instability of the upright position and makes the stabilization task more difficult, as you would experience when you try to balance a stick in your hand.

**Predicting System Behavior from Eigenvalue Locations**

We have seen that the solutions of a discrete-time system are composed of $\lambda_i^t$ terms where $\lambda_i$’s are the eigenvalues of $A$. Thus, to predict the nature of the solutions (damped, underdamped, unbounded, etc.), it is important to visualize the sequence $\lambda_i^t$, $t = 1, 2, \ldots$ for a given $\lambda$. If we rewrite $\lambda$ as $\lambda = |\lambda|e^{j\omega}$ where $|\lambda|$ is the distance to the origin in the complex plane, then we get

$$\lambda^t = |\lambda|^t e^{j\omega t} = |\lambda|^t \cos(\omega t) + j|\lambda|^t \sin(\omega t),$$

the real part of which is depicted in Figure 1 for various values of $\lambda$. Note that the envelope $|\lambda|^t$ decays to zero when $\lambda$ is inside the unit disk ($|\lambda| < 1$) and grows unbounded when it is outside ($|\lambda| > 1$), which is consistent with our stability criterion.

Likewise, for a continuous-time system each eigenvalue $\lambda_i$ contributes a function of the form $e^{\lambda_i t}$ to the solution. Decomposing $\lambda$ into its real and imaginary parts, $\lambda = v + j\omega$, we get

$$e^{\lambda t} = e^{vt}e^{j\omega t} = e^{vt} \cos(\omega t) + j e^{vt} \sin(\omega t).$$

Figure 2 depicts the real part of $e^{\lambda t}$ for various values of $\lambda$. Note that the envelope $e^{vt}$ decays when $v = \text{Re}(\lambda) < 0$ as in our stability condition.

**Example:**

In Lecture 4A we modeled the RLC circuit depicted on the right as

$$\begin{align*}
\frac{dx_1(t)}{dt} &= \frac{1}{C}x_2(t) \\
\frac{dx_2(t)}{dt} &= \frac{1}{L} (-x_1(t) - Rx_2(t) + u(t))
\end{align*}$$

where $x_1 = v_C$ and $x_2 = i$. Since this model is linear we can rewrite it in the form (2), with

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}.$$
Then the roots of
\[ \det(\lambda I - A) = \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} \]
give the eigenvalues:
\[ \lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \quad \text{where} \quad \alpha \triangleq \frac{R}{2L}, \quad \omega_0 \triangleq \frac{1}{\sqrt{LC}}. \]

For \( \alpha > \omega_0 \) we have two real, negative eigenvalues which indicate a damped response. For \( \alpha < \omega_0 \), we get the complex eigenvalues
\[ \lambda_{1,2} = -\alpha \mp j\omega \quad \text{where} \quad \omega \triangleq \sqrt{\omega_0^2 - \alpha^2}, \]
indicating oscillations with frequency \( \omega \) and decaying envelope \( e^{-\alpha t} \).

In subsequent lectures we will see that we can move the eigenvalues of a system to desired locations with feedback design. Therefore, it is useful to study Figures 1 and 2, and be able to associate eigenvalue locations with the types of system solutions that result.

Figure 1: The real part of \( \lambda^t \) for various values of \( \lambda \) in the complex plane. It grows unbounded when \( |\lambda| > 1 \), decays to zero when \( |\lambda| < 1 \), and has constant amplitude when \( \lambda \) is on the unit circle \( (|\lambda| = 1) \).
Figure 2: The real part of $e^{\lambda t}$ for various values of $\lambda$ in the complex plane. Note that $e^{\lambda t}$ is oscillatory when $\lambda$ has an imaginary component. It grows unbounded when $\Re\{\lambda\} > 0$, decays to zero when $\Re\{\lambda\} < 0$, and has constant amplitude when $\Re\{\lambda\} = 0$. 