EE16B - Fall'16 - Lecture 5B Notes[1]

Murat Arcak

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Stability of Linear State Models Continued

We have seen that the discrete time system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \tag{1}$$

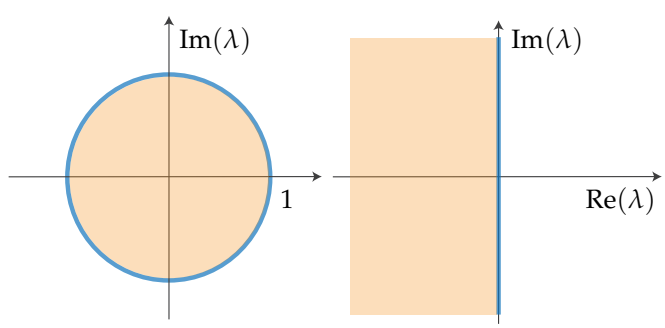
is stable if $|\lambda_i| < 1$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of A, and unstable if $|\lambda_i| > 1$ for some eigenvalue λ_i.

The continuous time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{2}$$

is stable if $\text{Re}\{\lambda_i\} < 0$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of A, and unstable if $\text{Re}\{\lambda_i\} > 0$ for some eigenvalue λ_i.

The figures below highlight the regions of the complex plane where the eigenvalues must lie for stability of a discrete-time (left) and continuous-time (right) system.



<u>Example:</u> In Lecture 4B we derived continuous-time linearized models for the downward and upright positions of the pendulum, and obtained:

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & \frac{-k}{m} \end{bmatrix} \qquad A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-k}{m} \end{bmatrix}. \tag{3}$$

The eigenvalues of A_{down} are the roots of $\lambda^2 + \frac{k}{m}\lambda + \frac{g}{\ell}$, which can be shown to have strictly negative real parts when $k > 0$. Thus the downward position is stable.

The eigenvalues of A_{up} are the roots of $\lambda^2 + \frac{k}{m}\lambda - \frac{g}{\ell}$, which are given by:

$$\lambda_1 = -\frac{k}{2m} - \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{\ell}}, \quad \lambda_2 = -\frac{k}{2m} + \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + 4\frac{g}{\ell}}.$$

Since $\lambda_2 > 0$, the upright position in unstable. Note that making the length ℓ smaller increases the value of λ_2. This suggests that a smaller length aggravates the instability of the upright position and makes the stabilization task more difficult, as you would experience when you try to balance a stick in your hand.

Predicting System Behavior from Eigenvalue Locations

We have seen that the solutions of a discrete-time system are composed of λ_i^t terms where λ_i's are the eigenvalues of A. Thus, to predict the nature of the solutions (damped, underdamped, unbounded, etc.), it is important to visualize the sequence $\lambda^t,\ t = 1, 2, \ldots$ for a given λ. If we rewrite λ as $\lambda = |\lambda| e^{j\omega}$ where $|\lambda|$ is the distance to the origin in the complex plane, then we get

$$\lambda^t = |\lambda|^t e^{j\omega t} = |\lambda|^t \cos(\omega t) + j\,|\lambda|^t \sin(\omega t),$$

the real part of which is depicted in Figure 1 for various values of λ. Note that the envelope $|\lambda|^t$ decays to zero when λ is inside the unit disk ($|\lambda| < 1$) and grows unbounded when it is outside ($|\lambda| > 1$), which is consistent with our stability criterion.

Likewise, for a continuous-time system each eigenvalue λ_i contributes a function of the form $e^{\lambda_i t}$ to the solution. Decomposing λ into its real and imaginary parts, $\lambda = v + j\omega$, we get

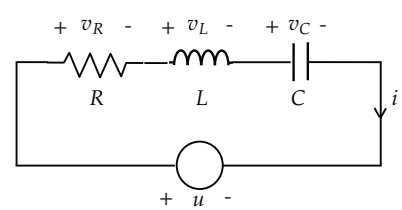
$$e^{\lambda t} = e^{vt} e^{j\omega t} = e^{vt} \cos(\omega t) + j\, e^{vt} \sin(\omega t).$$

Figure 2 depicts the real part of $e^{\lambda t}$ for various values of λ. Note that the envelope e^{vt} decays when $v = \text{Re}(\lambda) < 0$ as in our stability condition.

<u>Example:</u>

In Lecture 4A we modeled the RLC circuit depicted on the right as

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \frac{1}{C} x_2(t) \\ \frac{dx_2(t)}{dt} &= \frac{1}{L}\left(-x_1(t) - R x_2(t) + u(t)\right) \end{aligned} \tag{4}$$



where $x_1 = v_C$ and $x_2 = i$. Since this model is linear we can rewrite it in the form (2), with

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}.$$

Then the roots of

$$\det(\lambda I - A) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

give the eigenvalues:

$$\lambda_{1,2} = -\alpha \mp \sqrt{\alpha^2 - \omega_0^2} \quad \text{where} \quad \alpha \triangleq \frac{R}{2L},\ \omega_0 \triangleq \frac{1}{\sqrt{LC}}.$$

For $\alpha > \omega_0$ we have two real, negative eigenvalues which indicate a damped response. For $\alpha < \omega_0$, we get the complex eigenvalues

$$\lambda_{1,2} = -\alpha \mp j\omega \quad \text{where} \quad \omega \triangleq \sqrt{\omega_0^2 - \alpha^2},$$

indicating oscillations with frequency ω and decaying envelope $e^{-\alpha t}$.

In subsequent lectures we will see that we can move the eigenvalues of a system to desired locations with feedback design. Therefore, it is useful to study Figures 1 and 2, and be able to associate eigenvalue locations with the types of system solutions that result.

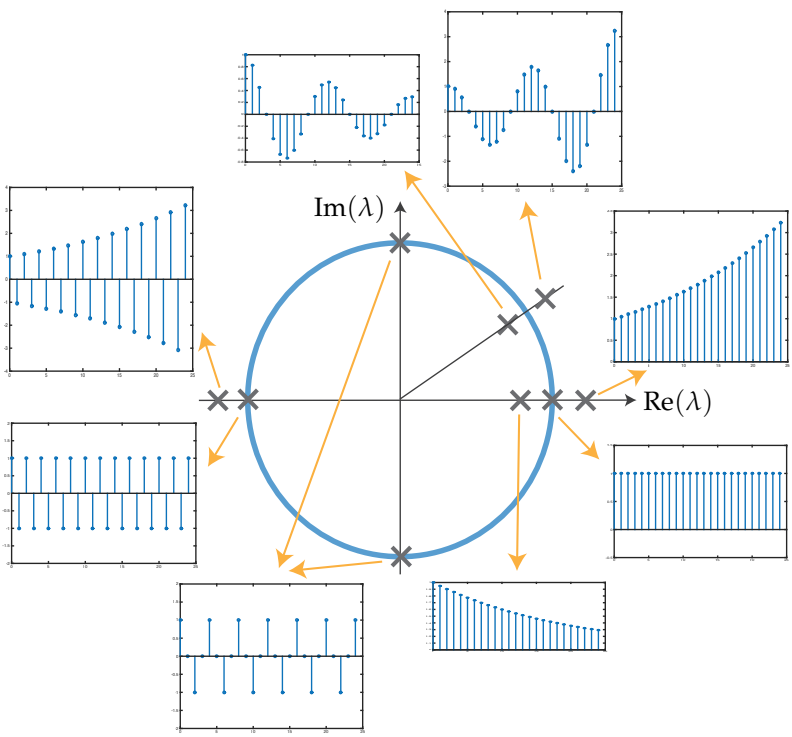


Figure 1: The real part of λ^t for various values of λ in the complex plane. It grows unbounded when $|\lambda| > 1$, decays to zero when $|\lambda| < 1$, and has constant amplitude when λ is on the unit circle ($|\lambda| = 1$).

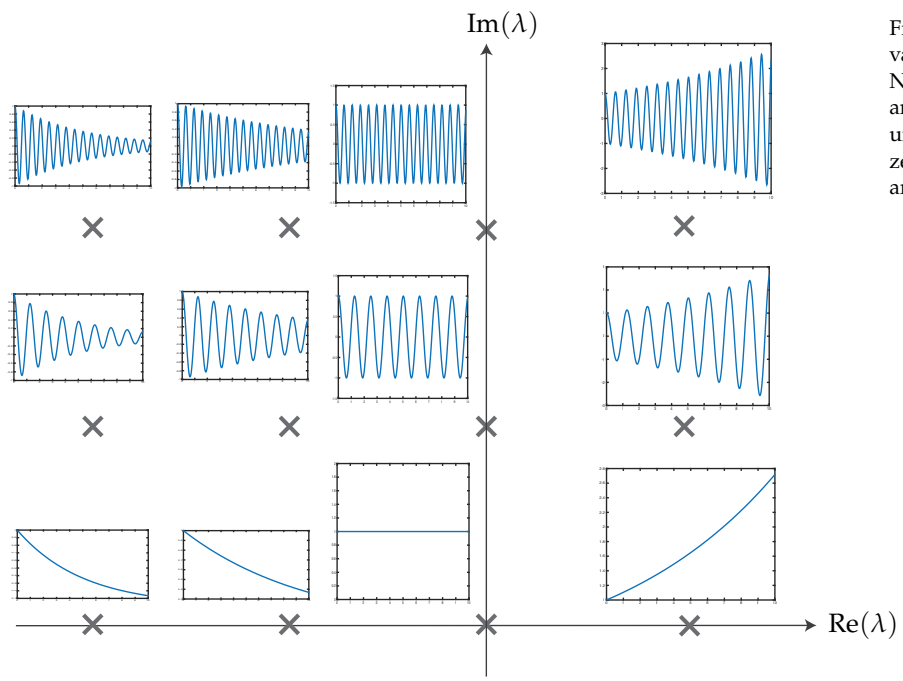


Figure 2: The real part of $e^{\lambda t}$ for various values of λ in the complex plane. Note that $e^{\lambda t}$ is oscillatory when λ has an imaginary component. It grows unbounded when $\text{Re}\{\lambda\} > 0$, decays to zero when $\text{Re}\{\lambda\} < 0$, and has constant amplitude when $\text{Re}\{\lambda\} = 0$.