State Feedback Control

Suppose we are given a single-input control system

\[ \vec{x}(t+1) = A\vec{x}(t) + Bu(t), \quad u(t) \in \mathbb{R}, \quad (1) \]

and we wish to bring the solution \( \vec{x}(t) \) to the equilibrium \( \vec{x} = 0 \) from any initial condition \( \vec{x}(0) \).

To achieve this goal we will study a “control policy” of the form

\[ u(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t) \quad (2) \]

where \( k_1, k_2, \ldots, k_n \) are to be determined. Rewriting (2) as

\[ u(t) = K \vec{x}(t) \quad (3) \]

with row vector \( K = [k_1 k_2 \cdots k_n] \), and substituting in (1), we get

\[ \vec{x}(t+1) = (A + BK)\vec{x}(t). \quad (4) \]

Thus, if we can choose \( K \) such that all eigenvalues of \( A + BK \) satisfy the stability condition \( |\lambda_i(A + BK)| < 1 \), then \( \vec{x}(t) \to 0 \) from any \( \vec{x}(0) \).

We will see in the next lecture that if the system (1) is controllable, then we can arbitrarily assign the eigenvalues of \( A + BK \) with the choice of \( K \). Thus, in addition to bringing the eigenvalues inside the unit disk for stability, we can place them in favorable locations to shape the transients, e.g., to achieve a well damped convergence.

We refer to (4) as the "closed-loop" system since the control policy (2) generates a feedback loop as depicted in the block diagram. The state variables are measured at every time step \( t \) and the input \( u(t) \) is synthesized as a linear combination of these measurements.
Comparison to Open Loop Control

Recall from the last lecture that controllability allows us to calculate an input sequence $u(0), u(1), u(2), \ldots$ that drives the state from $\vec{x}(0)$ to any $\vec{x}_{\text{target}}$. Thus, an alternative to the feedback control (2) is to select $\vec{x}_{\text{target}} = 0$, calculate an input sequence based on $\vec{x}(0)$, and to apply this sequence in an “open-loop” fashion without using further state measurements as depicted below.

\[
\begin{align*}
u(0), u(1), u(2), \ldots & \quad \rightarrow \\
\vec{x}(t+1) &= A\vec{x}(t) + Bu(t)
\end{align*}
\]

The trouble with this open-loop approach is that it is sensitive to uncertainties in $A$ and $B$, and does not make provisions against disturbances that may act on the system.

By contrast, feedback offers a degree of robustness: if our design of $K$ brings the eigenvalues of $A + BK$ to well within the unit disk, then small perturbations in $A$ and $B$ would not move these eigenvalues outside the disk. Thus, despite the uncertainty, solutions converge to $\vec{x} = 0$ in the absence of disturbances and remain bounded in the presence of bounded disturbances.

Example (Cruise Control): Consider again a vehicle moving in a lane (Lecture 6A) where the state equation for the velocity $v(t)$ is

\[
v(t+1) = v(t) + Tu(t). \quad (5)
\]

Suppose we want to stabilize the velocity to a desired value $v^*$. Define $\tilde{v}(t) = v(t) - v^*$, and subtract $v^*$ from both sides of the equation:

\[
\tilde{v}(t+1) = \tilde{v}(t) + Tu(t). \quad (6)
\]

Then the controller

\[
u(t) = k\tilde{v}(t) = k(v(t) - v^*)
\]

results in the closed-loop system

\[
\tilde{v}(t+1) = (1 + kT)\tilde{v}(t)
\]

which is stable if we choose the coefficient $k$ such that $|1 + kT| < 1$. 


Eigenvalue Assignment: Second Order Examples

Example: Consider the second order system
\[ \ddot{x}(t + 1) = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \]
\[ A \] \[ B \] \[ \text{Equation 7} \]

and note that the eigenvalues of \( A \) are the roots of the polynomial\(^1\) obtained from \( \det(\lambda I - A) \)
\[ \lambda^2 - a_2 \lambda - a_1. \]

If we substitute the control
\[ u(t) = K\ddot{x}(t) = k_1x_1(t) + k_2x_2(t) \]

the closed-loop system becomes
\[ \ddot{x}(t + 1) = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix} \ddot{x}(t) \]
\[ \begin{array}{c} A \\ +BK \end{array} \] \[ \text{Equation 8} \]

and, since \( A + BK \) has the same structure as \( A \) with \( a_1, a_2 \) replaced by \( a_1 + k_1, a_2 + k_2 \), the eigenvalues of \( A + BK \) are the roots of
\[ \lambda^2 - (a_2 + k_2) \lambda - (a_1 + k_1). \]

Now if we want to assign the eigenvalues of \( A + BK \) to desired values \( \lambda_1 \) and \( \lambda_2 \), we must match the polynomial above to
\[ (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2) \lambda + \lambda_1 \lambda_2, \]
that is,
\[ a_2 + k_2 = \lambda_1 + \lambda_2 \quad \text{and} \quad a_1 + k_1 = -\lambda_1 \lambda_2. \]

This is indeed accomplished with the choice \( k_1 = -a_1 - \lambda_1 \lambda_2 \) and \( k_2 = -a_2 + \lambda_1 + \lambda_2 \), which means that we can assign the closed-loop eigenvalues as we wish.

Example: Let’s apply the eigenvalue assignment procedure above to
\[ \ddot{x}(t + 1) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \]
\[ A \] \[ B \]

Now we have
\[ A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 + k_1 & 1 + k_2 \\ 0 & 2 \end{bmatrix} \]
and, because this matrix is upper diagonal, its eigenvalues are the diagonal entries:

\[ \lambda_1 = 1 + k_1 \quad \text{and} \quad \lambda_2 = 2. \]

Note that we can move \( \lambda_1 \) with the choice of \( k_1 \), but we have no control over \( \lambda_2 \). In fact, since \(|\lambda_2| > 1\), the closed-loop system remains unstable no matter what control we apply.

This is a consequence of the uncontrollability of this example which was shown in the previous lecture. The second state equation

\[ x_2(t + 1) = 2x_2(t) \]

can’t be influenced by \( u(t) \), and \( x_2(t) = 2^t x_2(0) \) grows exponentially.

By contrast the previous example was controllable\(^3\). In the next section we argue that controllability allows us to arbitrarily assign the eigenvalues of \( A + BK \) with the choice of \( K \).

\(^3\) Show this.