1. Otto the Pilot

Otto has devised a control algorithm so that his plane climbs to the desired altitude by itself. However he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \mp j\omega$$

then his altitude would indeed oscillate with frequency $\omega$ about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$, near its maxima and minima respectively.

![Altitude plot with curves](image)

(a) Find the real part $v$ and the imaginary part $\omega$ from the altitude plot.

**Solution:** Solving $1 + e^{5v} = 1.4843$ gives us $v = -0.1450 \, \text{min}^{-1}$. Then, comparing the maxima that are separated by an interval of 10 minutes gives $\omega = \frac{2\pi}{10} = 0.62832 \, \text{rad/min}$.

If you solved in units of $s^{-1}$ and rad/s, then $v = -0.0024 \, s^{-1}$ and $\omega = 0.0105 \, \text{rad/s}$.

(b) Let the dynamical model for the altitude be

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and $a_1$ and $a_2$ are constants. Using your answer to part (a), find out what $a_1$ and $a_2$ are.

**Solution:** The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ are given by $0 = \lambda^2 - a_2\lambda - a_1$, or equivalently

$$\lambda = \frac{a_2 \mp \sqrt{a_2^2 + 4a_1}}{2} = v \mp j\omega.$$
Solving for $a_1$ and $a_2$ (using the min$^{-1}$ and rad/min values of $v$ and $\omega$) we get

$$a_2 = 2v = -0.2900$$
$$a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.$$

If you solved using the $s^{-1}$ and rad/min values of $v$ and $\omega$, then

$$a_2 = 2v = -0.0048$$
$$a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \times 10^{-4}.$$

(c) Otto can change $a_2$ by turning a knob. Tell him what value he should pick so that he has a “critically damped” ascent with two negative, real eigenvalues at the same location.

**Solution:** To get two real identical eigenvalues Otto should choose $a_2$ to make $a_2^2 + 4a_1 = 0$. This means $a_2 = \pm 2\sqrt{-a_1}$. Since $a_2$ must be negative for the system to be stable, we only look at the negative root.

Solving with the $a_1$ derived from the min$^{-1}$ and rad/min values of $v$ and $\omega$, he should tune his knob to

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897.$$

If you solved using $a_1$ derived from the $s^{-1}$ and rad/s values of $v$ and $\omega$, then you get

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \times 10^{-4}} = -0.0215.$$

2. **LED Strip**

I have an LED strip with 5 red LEDs whose brightnesses I want to set. These LEDs are addressed as a queue: at each time step, I can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

(a) What should we use for our state vector? What does it mean that this is a state vector? What is our input?

**Solution:** We can use the brightnesses of each LED as our state vector. We can use these values as our state vector since together with the input, they describe everything about our system that we need to know in order to predict what our system will do in the future. Our input is the command to the left-most LED.

(b) Is our system linear? If it is linear, write out the state equations in matrix form. Please choose a reasonable order for the state variables in the state vector.

**Solution:** The system is linear because it can be written in the form $\vec{x}(t+1) = A\vec{x}(t) + Bu$. Ordering the LED brightnesses in the state vector from left to right, we get:

$$\vec{x}(t+1) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\vec{x}(t) \\
u(t) \\
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
$$

If you chose to put the left-most LED’s brightness last in the state vector (so that the LEDs are ordered right to left and the state vector gets flipped upside down), the $A$ matrix gets transposed and the $B$ matrix is flipped upside down.
(c) Is this system controllable? Explain intuitively what this system’s controllability means in terms of LED brightnesses.

**Solution:** Testing for controllability we see:

\[
\begin{bmatrix}
A^4 & A^3 & A^2 & A & B
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

which has full rank. This means that the system is controllable. A system is called controllable if from any initial state, we can reach any final state that we desire at some time in the future.

For our LED strip, controllability means that we can display any set of brightnesses that we desire, but it may take a few time steps to get there.

(d) Is this system stable?

**Solution:** The eigenvalues of this discrete-time system are all 0, which is inside the unit circle. Therefore, the system is stable.

(e) Starting from the pattern of brightnesses (from left to right) \([0, 127, 0, 255, 0]\), can we maintain this pattern for all future time steps? Can we display any fixed pattern of brightnesses for all time?

**Solution:** We cannot display \([0, 127, 0, 255, 0]\) for all time. Immediately after we display this set of brightnesses, we will display \([u(1), 0, 127, 0, 255]\).

If we want to display a fixed and unchanging set of brightnesses, every element in our state vector must be the same.

Controllability tells us only that we can reach any desired state (sometimes only temporarily). It does not mean we can keep our system at any desired state for all time.

3. Controllability and discretization

In this problem, we will use the car model

\[
\begin{align*}
\frac{d}{dt} p(t) &= v(t) \\
\frac{d}{dt} v(t) &= u(t)
\end{align*}
\]

that was discussed in class.

(a) Assuming that the input \(u(t)\) can be varied continuously, is this system controllable?

**Solution:** Introducing states \(x_1 = p\) and \(x_2 = v\), we rewrite this system in state space form

\[
\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).
\]

The controllability matrix

\[
C = [B \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

has rank 2. Therefore the continuous-time system is controllable.
(b) Now assume that we can only change our control input every $T$ seconds. Derive a discrete-time state space model for the state updates, assuming that the input is held constant between times $t$ and $t+T$.

**Solution:** By integrating both sides of the second equation from $t$ to $t+T$ and keeping in mind that $u(t)$ is constant in this interval, that is $u(t + \tau) = u(t)$ for $\tau \in [0, T)$:

$$v(t + T) - v(t) = \int_0^T u(t + \tau) d\tau = Tu(t).$$

Now integrating the first equation and using the fact that $v(t + \tau) = v(t) + \tau u(t)$ we get

$$p(t + T) - p(t) = \int_0^T (v(t) + \tau u(t)) d\tau = T v(t) + \frac{1}{2} T^2 u(t).$$

Introducing states $x_1(k) = p(kT)$ and $x_2(k) = v(kT)$ we get the state space model

$$x(k + 1) = Ax(k) + Bu(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2} T^2 \\ T \end{bmatrix} u(k).$$

(c) Is the discrete-time system controllable?

**Solution:** The controllability matrix

$$\mathcal{C} = [B \ AB] = \begin{bmatrix} \frac{1}{2} T^2 & \frac{3}{2} T^2 \\ T & T \end{bmatrix}$$

has rank 2. So the discrete-time system is controllable.

4. Controllability in circuits

Consider the circuit in Figure 1 where $V_s$ is an input we can control:

![Circuit Diagram](image)

Figure 1: Controllability in circuits

(a) Write the state space model for this circuit.

**Solution:**

$$\frac{dI}{dt} = \frac{V_s - V_1 - V_2}{R} = C_1 \frac{dV_1}{dt} = C_2 \frac{dV_2}{dt}$$

$$\frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{RC_1} & \frac{1}{RC_1} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} V_s$$
(b) Show that this system is not controllable.

**Solution:** If we calculate $AB$, we find that it is a linear combination of $B$:

$$AB = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} + \frac{1}{RC_2} \\ \frac{1}{RC_1} & \frac{1}{RC_1} + \frac{1}{RC_2} \end{bmatrix} = -(\frac{1}{RC_1} + \frac{1}{RC_2})B$$

This means that the controllability matrix

$$\begin{bmatrix} B & AB \end{bmatrix}$$

Must have rank of 1. Therefore, this system is not controllable.

(c) Explain, in terms of circuit currents and voltages, why this system isn’t controllable. (Hint: think about what currents/voltages of the circuit we are controlling with $V_s$)

**Solution:** We can only control $V_s$, which in turn controls the amount of current flowing through the circuit. Since this current is equal through both capacitors and current directly affects the voltage across a capacitor, there is no way to individually control the voltages across the capacitors.

(d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

**Solution:**

We can control $V_3$ in this circuit.

5. **Controllability in 2D**

Consider the control of some two-dimensional linear discrete-time system

$$\tilde{x}(k+1) = A\tilde{x}(k) + Bu(k)$$

where $A$ is a $2 \times 2$ real matrix and $B$ is a $2 \times 1$ real vector.

(a) Let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ with $a, c, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$. Find a $B$ such that the system is controllable no matter what nonzero values $a, c, d$ take on, and a $B$ for which it is not controllable no matter what nonzero values are given for $a, c, d$. You can use the controllability rank test, but please explain your intuition as well.

**Solution:** With $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the system is controllable for all nonzeros $a, c, d$, because $[B, AB] = \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}$, which has full rank. With $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the system is not controllable because $[B, AB] = \begin{bmatrix} 0 & 0 \\ 1 & d \end{bmatrix}$, which
only has rank=1. The intuition is that, due to the zero entry in A, the state $x_1$ evolves autonomously, i.e., \( \frac{d}{dt} x_1(t) = ax_1(t) \), hence it needs to be controlled by some input $f$. On the other hand we can control $x_2$ via controlling $x_1$, as \( \frac{d}{dt} x_2(t) = cx_1(t) + dx_2(t) \), which implies that $x_2$ can be “tuned” by manipulating $x_1$.

(b) Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$. and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero $a, d, f, g$ that make the system uncontrollable. 

**Solution:** No. uncontrollable when $a = d$. In this case the matrix is just a constant $a$ times the identity. So when you check with the controllability test, $AB$ is just a scalar multiple of $B$ and hence linearly dependent. The intuition is that the two states are inherently “coupled” as two eigenvalues are the same. Any control input can only move the states along a line hence the states cannot reach arbitrary points in $\mathbb{R}^2$.

(c) We want to see if controllability is preserved under changes of coordinates. To begin with, let $\bar{z}(k) = V^{-1}x(k)$, please write out the system equation with respect to $\bar{z}$.

**Solution:** $\bar{x}(k) = V\bar{z}(k)$, hence we have

\[
V\bar{z}(k+1) = AV\bar{z}(k) + Bu(k)
\]

\[
\bar{z}(k+1) = V^{-1}AV\bar{z}(k) + V^{-1}Bu(k)
\]

(d) Now show that controllability is preserved under change of coordinates. (Hint: use the fact that \( \text{rank}(MA) = \text{rank}(A) \) for any invertible matrix $M$.)

**Solution:** The matrix whose rank needs to be tested after the coordinate change is $[V^{-1}B, V^{-1}AVV^{-1}B] = [V^{-1}B, V^{-1}AB] = V^{-1}[B, AB]$ which has the same rank as $[B, AB]$, since $V$ by assumption is full rank.

6. Understanding the SIXT33N car control model

As we continue along the process of making the SIXT33N cars be awesome, we’d like to better understand the car model which we will be using to develop a control scheme. As a wheel on the car turns, there is an encoder disc (see below) which also turns as the wheel turns. The encoder shines a light though the encoder disc, and as the wheel turns, the light is continually blocked and unblocked, allowing the encoder to detect rotations of the wheel per ‘tick’ of the encoder disc.

![Encoder disc](image)

The following model applies separately to each motor of the car:

\[
v(t) = d(t+1) - d(t) = \theta u(t) - \beta
\]

Meet the variables at play in this model:
• $t$ - The current timestep of the model. Since this is a discrete system, this will advance by 1 on every new sample in the system.

• $d(t)$ - The current number of ticks advanced by this wheel.

• $v(t)$ - The discrete-time velocity (in units of ticks/timestep) of the wheel, measured by finding the difference between two adjacent tick counts $(d(t+1) − d(t))$.

• $u(t)$ - The input to the system, in terms of "PWMs". As we will observe in lab, the circuit driving the model controls the amount of power delivered to the wheel in units of PWM intensity. This is a number between 0 and 255, where 0 means no power is delivered to the motor and 255 means maximum capable power is delivered to the motor.

• $\theta$ - In units of "ticks/PWM", this models how much more the motor turns for every increase in PWM. This is empirically measured from the car.

• $\beta$ - In units of ticks/timestep, this models the effect of friction on the car (if no power is applied to the motors, we’d expect the velocities of the motors to decrease). This is empirically measured from the car.

In this problem, we will assume that the motor conforms perfectly to this model to get an intuition of how the model works.

(a) If we wanted to make the motor drive at a certain target velocity $v^*$, with what PWM $u(t)$ should we feed the motor?

**Solution:**

\[
\begin{align*}
v^* &= \theta u(t) - \beta \\
v^* + \beta &= \theta u(t) \\
u(t) &= \frac{v^* + \beta}{\theta}
\end{align*}
\]

(b) What signs should $\theta$ and $\beta$ have? Should they be positive or negative? Note that applying PWM to the motor driver circuit can only ever deliver power in a way so as to cause the motor to move forwards and never backwards, and there are no braking mechanisms on the motor.

**Solution:** $\theta > 0$ since applying PWM should only ever increase the velocity. If friction is a large factor in the linear fit for our motor model, we expect that $\beta$ should be greater than zero since friction should decrease the speed of the car in the absence of PWM input. However, nonlinearities and imperfections in our motors may outweigh the effect of friction so that $\beta$ may experimentally end up positive.

(c) Even if the motor conforms perfectly to the model, our inputs still limit the range of velocities of the motor. Given that $0 \leq u(t) \leq 255$\(^1\) determine the maximum and minimum velocities possible with the motor. What does this tell us about the braking of the car?

**Solution:** The maximum is $\theta 255 - \beta$ and the minimum is $0 - \beta = -\beta$.

Since there are no brakes on the motor, we slow down by reducing the PWM.

(d) Our intuition tells us that a motor on a car should eventually stop turning if we stop applying any power to it. Find $v(t)$ as $t \to \infty$, assuming $v(0) = v_0$ (say $v_0 > 0$) and $u(t) = 0$. Does our model obey our intuition? What does that tell us about our model?

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\(^1\)See [https://www.arduino.cc/en/uploads/Tutorial/pwm1.gif](https://www.arduino.cc/en/uploads/Tutorial/pwm1.gif) for an example of how PWM works and why this is the case.
Solution:

\[ v(\infty) = -\beta \]

However our intuition says that the motor should have stopped: \( v(\infty) = 0 \). In lab we empirically find the value of \( \beta \) over a range of PWM values, but our fit does not work very well everywhere and our model does not match the real behavior near \( u = 0 \).

Stay tuned for closed-loop control of the SIXT33N motors!

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