Designing Information Devices and Systems II

Lecture 5B
Linearization
Stability of linear state models
Intro

• Last time
  − Described systems with state-space model
  − Talked about linear systems
  − Change of variables

• Today
  − Linearization of non-linear systems
  − Begin Stability of linear state models
    • Scalar and discrete
Linearization

State variables:

\[ x_1(t) = \theta(t) \]
\[ x_2(t) = \dot{\theta}(t) = \frac{d\theta}{dt} \]

\[ \frac{dx_1(t)}{dt} = x_2(t) \]
\[ \frac{dx_2(t)}{dt} = -\frac{g}{l} \sin(x_1(t)) - \frac{k}{m} x_2(t) \]

Linearization:

\[ \frac{dx_2(t)}{dt} = -\frac{g}{l} x_1(t) - \frac{k}{m} x_2(t) \]
Linearization

\[
\frac{dx_1(t)}{dt} = x_2(t)
\]

\[
\frac{dx_2(t)}{dt} = -\frac{g}{l} x_1(t) - \frac{k}{m} x_2(t)
\]

\[
\Rightarrow \begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{g}{l} & -\frac{k}{m}
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\]
Scary Example: Pole on a Cart

How to systematically linearize?

\[
\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2 \theta} \left( \frac{u}{m} + \dot{\theta}^2 l \sin \theta - g \sin \theta \cos \theta \right)
\]

\[
\ddot{\theta} = \frac{1}{l\left(\frac{M}{m} + \sin^2 \theta\right)} \left( -\frac{u}{m} \cos \theta - \dot{\theta}^2 l \sin \theta \cos \theta + \frac{M + m}{m} g \sin \theta \right)
\]
Taylor Approximation - scalar

\( f : \mathbb{R} \rightarrow \mathbb{R} \)

\[
f(x) \approx f(x^*) + f'(x^*)(x - x^*)
\]

\[
\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)
\]

\[
x^* = 0 \quad \Rightarrow \sin(x) \approx \sin(0) + \cos(0)(x - 0)
\]

\[
\sin x \approx x
\]
Taylor Approximation - vector

\[ f : \mathbb{R}^N \rightarrow \mathbb{R}^N \]

\[ \frac{d}{dt} \vec{x} = f(\vec{x}) \]

\[ f(\vec{x}) \approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*) \]

\[ \text{Q: What are the dimensions of } \nabla f(\vec{x}^*) \text{? (Jacobian)} \]
Taylor Approximation - vector

\[ f : \mathbb{R}^N \rightarrow \mathbb{R}^N \]

\[ f(\vec{x}) = \begin{bmatrix}
  f_1(x_1, \ldots, x_N) \\
  f_2(x_1, \ldots, x_N) \\
  \vdots \\
  f_N(x_1, \ldots, x_N)
\end{bmatrix} \]

\[ \nabla f(\vec{x}) = \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N}
\end{bmatrix} \]

i,j^{th} entry:

\[ \frac{\partial f_i(x)}{\partial x_j} \]
Taylor Approximation - vector

\[
f : \mathbb{R}^N \rightarrow \mathbb{R}^N \quad f(\bar{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\
                        f_2(x_1, \cdots, x_N) \\
                        \vdots \\
                        f_N(x_1, \cdots, x_N) \end{bmatrix}
\]

\[
\nabla f(\bar{x}) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N}
\end{bmatrix}
\]

i,j \textsuperscript{th} entry: \[
\frac{\partial f_i(x)}{\partial x_j}
\]
Linearization of State-Space

Linearize around an equilibrium, a point s.t.:

\[ f(\vec{x}^*) = 0 \]

\[ \frac{d}{dt} \vec{x} = f(\vec{x}) \]

\[ \approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*) \]

\[ = 0 \]

Which of the variable is a function of t?

write a state model for deviation!
Linearization of State-Space

\[ \tilde{x} = \bar{x} - \bar{x}^* \]

\[
\frac{d}{dt} \tilde{x}(t) = \frac{d}{dt} \bar{x}(t) - \frac{d}{dt} \bar{x}^* = 0
\]

\[
= f(\bar{x}(t)) \approx f(\bar{x}^*) + \nabla f(\bar{x}^*) \tilde{x}
\]

\[
\frac{d}{dt} \tilde{x}(t) = [\nabla f(\bar{x}^*)] \tilde{x}(t)
\]
Back to the Pendulum

\[ f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l} \sin(x_1(t)) - \frac{k}{m} x_2(t) \end{bmatrix} \]

\[ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos x_1 & -\frac{k}{m} \end{bmatrix} \]
Back to the Pendulum

\[ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos x_1 & -\frac{k}{m} \end{bmatrix} \]

\( x_1^* = 0, x_2^* = 0 \), Downward equilibrium

\[ A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \]

This is small signal analysis!

\( x_1^* = \pi, x_2^* = 0 \), Upward equilibrium

\[ A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} \]

Talk about next lecture!
Discrete Time

\[ \ddot{x}(t + 1) = f(\ddot{x}(t)) \]

\( \ddot{x} = \ddot{x}^* \) is an equilibrium if:

\[ f(\ddot{x}^*) = \ddot{x}^* \]

(for cont. \( f(\ddot{x}^*) = 0 \))

\[ \tilde{x}(t) = \ddot{x}(t) - \ddot{x}^* \]

\[ \tilde{x}(t + 1) = \ddot{x}(t + 1) - \ddot{x}^* \]

\[ = f(\ddot{x}(t)) - \ddot{x}^* \]

\[ \approx f(\ddot{x}^*) + \nabla f(\ddot{x}^*) \tilde{x}(t) - \ddot{x}^* \]

\[ \tilde{x}(t + 1) = A\tilde{x}(t) \]
Stability of Linear State Models

Start with scalar system:

\[ x(t + 1) = ax(t + 1) + bu(t) \]

Given initial condition \( x(0) \):

\[ x(1) = ax(0) + bu(0) \]

\[ x(2) = ax(1) + bu(1) = a^2x(0) + abu(0) + bu(1) \]

\[ x(3) = a^3x(0) + a^2bu(0) + abu(1) + bu(2) \]

\[ x(t) = a^tx(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots + a^0bu(t-1) \]
Stability of Linear State Models

Start with scalar system:

\[ x(t + 1) = ax(t + 1) + bu(t) \]

Given initial condition \( x(0) \):

\[ x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \]

- Initial condition
- input
Stability - Definition

• A system is stable if $\bar{x}(t)$ is bounded for any initial condition $\bar{x}(0)$ and any bounded input sequence $u(0), u(1), \cdots$

• A system is unstable if there is an $\bar{x}(0)$ or a bounded input sequence for which

$$|\bar{x}(t)| \to \infty \quad \text{as} \quad t \to \infty$$
Example

Q) Is this system stable?

\[ x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} u(k) \]
Stability Proof

Claim 1: if $|a| < 1$ then the system is stable

Proof: $a^t \to 0$ as $t \to \infty$ because $|a| < 1$ so, initial condition always bounded.

Sequence is bounded – there exists $M$ s.t. $|u(t)| \leq M \forall t$

$$x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k)$$
Stability Proof Cont.

\[
\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| |u(k)| \leq M
\]

Define: \( s = t - k - 1 \)

\[
= \sum_{s=0}^{t-1} |a|^s |b| M = |b| M \sum_{s=0}^{t-1} |a|^s \leq |b| M \frac{1}{1 - |a|}
\]

\[
\sum_{s=0}^{\infty} |a|^s = \frac{1}{1 - |a|}, \quad |a| < 1
\]
Stability Proof Cont.

Claim 2: unstable when $|a| > 1$

Proof: if $x(0) \neq 0$ (even $u(t)=0 \ \forall \ t$)

$$x(t) = a^t x(0) \rightarrow \infty$$

Q: What if $|a| = 1$, i.e., $a=1$ or $a=-1$
Quiz

With input $u(t)=M$, $a=-1$

\[
\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq b M
\]

Q: what $|u(t)| \leq M$ will make it unstable?
Stability Cont.

What if $a$ is complex valued?

$$|a| < 1 \implies \text{stable}$$

$$|a| \geq 1 \implies \text{unstable}$$

$$|a| = \sqrt{\text{Re}(a)^2 + \text{Im}(a)^2}$$
Summary

• Described linearization about an equilibrium point
  – Continuous time
  – Discrete time

• Conditions for stability of a linear system
  – Discrete
  – First order
  – Scalar

• Next time:
  – Vector case! (which leads to Eigen-value analysis)