Linearization of systems

One dimensional linear approximation

Consider a differentiable function $f$ of one variable.

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Say we are interested in $f$ in a small, open neighborhood about a particular point $t^*$. $t^*$ is often called the fixed point of the system. Let’s call this open neighborhood $U$. In this case, we can construct a linear approximation of $f$ about the neighborhood $U$. Recall Taylor Approximation,

$$\frac{df}{dt}(t^*) \approx \frac{f(t) - f(t^*)}{t - t^*} \quad \text{for } t \in U$$

We can use the above to construct a linear approximation of $f$. Let $f_i$ denote the linear approximation of $f$ about $U$.

$$f_i(t) = \frac{df}{dt}(t^*)(t - t^*) + f(t^*) \quad (1)$$

Strictly speaking, $f_i$ is an affine approximation unless $t^* = 0$, but the process of obtaining $f_i$ is colloquially called the linearization of $f$.

For example, consider $f(t) = t^2$. We will set the fixed point of the system to be $t^* = 1$. Then,

$$f_i(t) = \frac{df}{dt}(t^*)(t - t^*) + f(t^*)$$

$$= 2(t - 1) + 1$$

$$= 2t - 1$$

Let $\varepsilon = 10^{-2}$. Consider the open neighborhood $U = (1 - \varepsilon, 1 + \varepsilon)$. Let’s plot $f$ and $f_i$ when their respective domains are restricted to $U$. This is seen in Figure 1 on the next page.
Let’s define $\tilde{x} = x - \bar{x}, \tilde{u} = u - \bar{u}$. $\nabla f$ is the multidimensional generalization of the derivative, which is constructed as follows.

$$
\nabla_x f(x,u) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(x,u) & \frac{\partial f_1}{\partial x_2}(x,u) & \cdots & \frac{\partial f_1}{\partial x_n}(x,u) \\
\frac{\partial f_2}{\partial x_1}(x,u) & \frac{\partial f_2}{\partial x_2}(x,u) & \cdots & \frac{\partial f_2}{\partial x_n}(x,u) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1}(x,u) & \frac{\partial f_m}{\partial x_2}(x,u) & \cdots & \frac{\partial f_m}{\partial x_n}(x,u)
\end{bmatrix}
$$

and

$$
\nabla_u f(x,u) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial u_1}(x,u) & \frac{\partial f_1}{\partial u_2}(x,u) & \cdots & \frac{\partial f_1}{\partial u_m}(x,u) \\
\frac{\partial f_2}{\partial u_1}(x,u) & \frac{\partial f_2}{\partial u_2}(x,u) & \cdots & \frac{\partial f_2}{\partial u_m}(x,u) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial u_1}(x,u) & \frac{\partial f_m}{\partial u_2}(x,u) & \cdots & \frac{\partial f_m}{\partial u_m}(x,u)
\end{bmatrix}
$$

Linearization of a system

Consider a continuous, non-linear system with state $\tilde{x}(t)$ (which is $n$ dimensional) and input $\tilde{u}(t)$ (which is $m$ dimensional) of the form,

$$
\frac{d\tilde{x}(t)}{dt} = f(\tilde{x}(t), \tilde{u}(t))
$$

To clarify and establish notation, $f$ is a function that takes in $\tilde{x}(t)$ and $\tilde{u}(t)$ and outputs an $n$ dimensional vector. $f_k(\tilde{x}(t), \tilde{u}(t))$ refers to the function that is the $k^{th}$ coordinate of the output of $f(\tilde{x}(t), \tilde{u}(t))$.

Let $\bar{x}(t)$ be the desired state trajectory and $\bar{u}^*(t)$ be the desired input. Analogous to (1), we will construct an affine estimate about a neighborhood around the fixed points.

$$
f(\bar{x}(t), \bar{u}(t)) \approx f(\bar{x}^*(t), \bar{u}^*(t)) + \nabla_{\tilde{x}} f(\bar{x}^*(t), \bar{u}^*(t))(\bar{x}(t) - \bar{x}^*(t)) + \nabla_{\tilde{u}} f(\bar{x}^*(t), \bar{u}^*(t))(\bar{u}(t) - \bar{u}^*(t))
$$

Let’s define $\tilde{x} = \bar{x} - \bar{x}^*, \tilde{u} = \bar{u} - \bar{u}^*$. $\nabla f$ is the multidimensional generalization of the derivative, which is constructed as follows.

$$
\nabla_x f(x,u) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(x,u) & \frac{\partial f_1}{\partial x_2}(x,u) & \cdots & \frac{\partial f_1}{\partial x_n}(x,u) \\
\frac{\partial f_2}{\partial x_1}(x,u) & \frac{\partial f_2}{\partial x_2}(x,u) & \cdots & \frac{\partial f_2}{\partial x_n}(x,u) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1}(x,u) & \frac{\partial f_m}{\partial x_2}(x,u) & \cdots & \frac{\partial f_m}{\partial x_n}(x,u)
\end{bmatrix}
$$

and

$$
\nabla_u f(x,u) = 
\begin{bmatrix}
\frac{\partial f_1}{\partial u_1}(x,u) & \frac{\partial f_1}{\partial u_2}(x,u) & \cdots & \frac{\partial f_1}{\partial u_m}(x,u) \\
\frac{\partial f_2}{\partial u_1}(x,u) & \frac{\partial f_2}{\partial u_2}(x,u) & \cdots & \frac{\partial f_2}{\partial u_m}(x,u) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial u_1}(x,u) & \frac{\partial f_m}{\partial u_2}(x,u) & \cdots & \frac{\partial f_m}{\partial u_m}(x,u)
\end{bmatrix}
$$

Figure 1: Left: The linear approximation of the function $f(t) = t^2$ about $t = 1$ is fairly accurate in the range $U = [0.99, 1.01]$. Right: Outside of this range, the approximation does not track the function very well.
Note the dimensions of the matrices. It must be notated that, when calculating \( \nabla_x f, \bar{u} \) is considered constant. Similarly, when calculating \( \nabla_u f, \bar{x} \) is considered constant.

To continue from (2), note that,

\[
\frac{dx}{dt}(t) = f(x(t), u(t)), \quad \frac{dx^*}{dt}(t) = f(x^*(t), u^*(t)) \quad \text{and} \quad \frac{dx}{dt}(t) = \frac{dx^*}{dt}(t) + \frac{d\bar{x}}{dt}(t)
\]

Plugging this back into (2), we get,

\[
\frac{dx^*}{dt}(t) + \frac{d\bar{x}}{dt}(t) \approx \frac{dx^*}{dt}(t) + \nabla_x f(x^*(t), u^*(t))\bar{x}(t) + \nabla_u f(x^*(t), u^*(t))\bar{u}(t)
\]

Thus, we get linearized version of our system.

\[
\frac{dx}{dt}(t) \approx \nabla_x f(x^*(t), u^*(t))\bar{x}(t) + \nabla_u f(x^*(t), u^*(t))\bar{u}(t) \quad (3)
\]

Note that, unlike the one dimensional linearization example, we are linearizing with respect to \( \bar{x} \) and \( \bar{u} \). Also, observe that our state variables are now the perturbations \( \bar{x}(t) \) and \( \bar{u}(t) \).

Questions

1. **Jacobian Warm-Up**

   Consider the following function \( f : \mathbb{R}^2 \mapsto \mathbb{R}^3 \)

   \[
f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1x_2 \\ x_1 \end{pmatrix}
   \]

   Calculate its Jacobian.

   **Answer:**

   \[
   \frac{df}{dx} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1x_2 \\ 1 & 0 \end{pmatrix}
   \]

2. **Linearization**

   Consider a mass attached to two springs:
We assume that each spring is linear with spring constant $k$ and resting length $X_0$. We want to build a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m} (y - X_0 \frac{y}{\sqrt{y^2 + a^2}})$.

(a) Write this model in state space form $\dot{x} = f(x)$.

**Answer:** We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\
-\frac{2k}{m} \left( x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}} \right)
\end{bmatrix}.
\]

(b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

**Answer:** We find the equilibrium by solving $0 = \dot{x} = f(x)$:

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix} = \begin{bmatrix}
x_2 \\
-\frac{2k}{m} \left( x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}} \right)
\end{bmatrix}.
\]

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

(c) Linearize your model about the equilibrium.

**Answer:**

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}_{x=(0,0)} = \begin{bmatrix}
0 & 1 \\
\left( 1 - X_0 \frac{a^2}{(y^2 + a^2)^{3/2}} \right) & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
\left( 1 - \frac{X_0}{a} \right) & 0
\end{bmatrix}_{x=(0,0)}
\]

So the linearized system is

\[
\dot{x} = \begin{bmatrix}
0 & 1 \\
\left( 1 - \frac{X_0}{a} \right) & 0
\end{bmatrix} x.
\]