Primer to Complex Linear Algebra

Complex Conjugates and Adjoints

The complex conjugate of a complex number \( z = a + bj = re^{i\theta} \) is defined as

\[ z^* = z^H = a - bj = re^{-i\theta} \]

Let \( \vec{z} \in \mathbb{C}^n \).

\[ \vec{z} = [z_1 \ z_2 \ \ldots \ z_n]^T \]

The conjugate transpose (or adjoint, or Hermitian transpose) of \( \vec{z} \) is defined as

\[ \vec{z}^* = \vec{z}^H = [z_1^* \ z_2^* \ \ldots \ z_n^*] \]

Inner Product Properties

An inner product on a complex vector space \( \mathbb{C}^n \) is a function such that the following all hold for \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n \)

\[ \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle^* \]

\[ \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \]

\[ \langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle \]

\[ \langle \vec{u}, \vec{u} \rangle \geq 0 \]

\[ \langle \vec{u}, \vec{u} \rangle = 0 \implies \vec{u} = \vec{0} \]

Questions

1. Controls

Consider the following system:

\[
\frac{dx_1(t)}{dt} = -x_1(t)^2 + x_2(t)u(t) \\
\frac{dx_2(t)}{dt} = 2x_1(t) - 2x_2(t)u(t)
\]
(a) Choose states and write a state space model for the system in the form $\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), u(t))$.

**Answer:**

Because the states should be related to their derivatives, we choose $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), u(t)) \\ f_2(\vec{x}(t), u(t)) \end{bmatrix} = \begin{bmatrix} -x_1(t)^2 + x_2(t)u(t) \\ 2x_1(t) - 2x_2(t)u(t) \end{bmatrix}$$

(b) Find the equilibrium $\vec{x}^*$ and input $u^*$ when $x_2^* = 1$ and $u^* = 1$.

**Answer:**

Plugging in $x_2^* = 1$ and $u^* = 1$, we solve the system of equations for $x_1^*$.

Looking at the second equation, we get:

$$0 = 2x_1^* - 2 \implies x_1^* = 1$$

$$\vec{x}^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u^* = 1$$

(c) Linearize the system around the equilibrium state and input from the previous part. Your answer should be in the form $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + B\vec{u}(t)$.

**Answer:**

Recall that $\vec{x}(t) \triangleq \vec{x}(t) - \vec{x}^*$ and $\vec{u}(t) \triangleq u(t) - u^*$.

From linearization, we get:

$$\frac{d\vec{x}(t)}{dt} = \left[ \begin{array}{c} \nabla_{\vec{x}} f(\vec{x}^*(t), u^*(t)) \vec{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t), u^*(t)) \vec{u}(t) \end{array} \right]$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2x_1^* & u^* \\ 2 & -2u^* \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} x_2^* \\ -2x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(d) Is this system controllable? Is it stable?

**Answer:**

$$R_2 = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = (-2 - \lambda)^2 - 2 = \lambda^2 + 4\lambda + 2 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 8}}{2} = -2 \pm \sqrt{2}$$

The controllability matrix has rank 2, so this system is controllable. This system also only has negative eigenvalues, so it is stable.
(e) Find a state feedback controller $K$ to place both system eigenvalues at $\lambda = -1$, where $\ddot{u}(t) = K\ddot{x}(t)$.

Answer:

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A + BK = \begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 + k_1 & 1 + k_2 \\ 2 - 2k_1 & 1 - 2k_2 \end{bmatrix}$$

Now, let’s find the characteristic polynomial for the closed-loop matrix:

$$\lambda^2 + \lambda(4 - k_1 + 2k_2) + (2 - k_1)(2k_2 + 2) - (1 + k_2)(2 - 2k_1) = 0$$

$$\lambda^2 + \lambda(4 - k_1 + 2k_2) + (2k_2 + 2) = 0$$

Next, comparing the coefficients of our required characteristic equation, $\lambda^2 + 2\lambda + 1 = 0$, we get:

$$4 + 2k_2 - k_1 = 2$$

$$2k_2 + 2 = 1$$

Solving the above equations, we get, $k_1 = 1$ and $k_2 = -\frac{1}{2}$.

Extra Practice

1. Feedback Design

Consider the following system:

$$\ddot{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\ddot{f}(\ddot{x}, u) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$\frac{dx_1(t)}{dt} = f_1(\ddot{x}, u) = x_1(t)^2x_2(t) - 4x_2(t) + u(t)x_2(t)$$

$$\frac{dx_2(t)}{dt} = f_2(\ddot{x}, u) = 2x_2(t) - 3x_1(t) - x_1(t)u(t)$$

(a) Find the equilibrium points of $\ddot{x}$ when $u(t) = 0$.

Answer:

To find the equilibrium points, we set the derivatives and $u(t)$ to 0:

$$0 = (x_1(t)^2 - 4)x_2(t) \quad (1)$$
0 = 2x_2(t) - 3x_1(t) \tag{2}

From Equation (2):

\[ x_2(t) = \frac{3}{2}x_1(t) \]

Plugging into Equation (1):

\[ 0 = (x_1(t)^2 - 4) \frac{3}{2}x_1(t) \]

\[ x_1(t) = -2, 0, 2 \]

With the \( x_1(t) \) values, we can find the corresponding \( x_2(t) \) values for each equilibrium point (in the form \((x_1, x_2)\)):

\[ \text{equilibrium points} = (-2, -3), (0, 0), (2, 3) \]

(b) Linearize the system around \( \vec{x}^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) and \( u^*(t) = 0 \).

**Answer:** Let, \( \vec{x} = \vec{x} - \vec{x}^* \) and \( \vec{u}(t) = u(t) - u^*(t) = u(t) \).

The linearized system will have the form:

\[
\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u}
\]

\[
\frac{d\vec{x}}{dt} = \nabla_x f(x, u)\vec{x} + \nabla_u f(x, u)\vec{u}\bigg|_{x_1=2, x_2=3, u=0}
\]

\[
\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \vec{x} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \vec{u}\bigg|_{x_1=2, x_2=3, u=0}
\]

\[
\frac{d\vec{x}}{dt} = \begin{bmatrix} 2x_1x_2 & x_1^2 - 4 + u \\ -3 - u & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \vec{u}\bigg|_{x_1=2, x_2=3, u=0}
\]

\[
\frac{d\vec{x}}{dt} = \begin{bmatrix} 12 & 0 \\ -3 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \vec{u}
\]

(c) Is the linearized system stable?

**Answer:**

To check stability, we need to find the eigenvalues of \( A \). Since \( A \) is lower triangular, the eigenvalues are \( \lambda = 2, 12 \). The condition for stability of a continuous system is \( \text{Re}\{\lambda\} < 0 \), so this system is unstable.

(d) Is the linearized system controllable?

**Answer:**

Since this is a 2 \( \times \) 2 system, the controllability matrix is:

\[
\mathcal{R}_2 = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 36 & 3 \\ -13 & -2 \end{bmatrix}
\]

The controllability matrix \( \mathcal{R}_2 \) is full rank, so the system is controllable.
(e) Using state feedback with \( \tilde{u} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \tilde{x} \), find \( k_1 \) and \( k_2 \) to make the system stable with \( \lambda = -1, -9 \).

**Answer:**

Plugging in \( \tilde{u}(t) \):

\[
\frac{d\tilde{x}}{dt} = \begin{bmatrix} 12 & 0 \\ -3 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tilde{x}
\]

\[
\frac{d\tilde{x}}{dt} = \begin{bmatrix} 12 + 3k_1 & 3k_2 \\ -3 - 2k_1 & 2 - 2k_2 \end{bmatrix} \tilde{x}
\]

\[
\det(A - \lambda I) = (12 + 3k_1 - \lambda)(2 - 2k_2 - \lambda) + 3k_2(3 + 2k_2) = 0
\]

\[
\lambda^2 - (3k_1 - 2k_2 + 14)\lambda + (6k_1 - 15k_2 + 24) = 0
\]

We need to set the characteristic polynomial to the characteristic polynomial with the desired eigenvalues:

\[
(\lambda + 9)(\lambda + 1) = \lambda^2 + 10\lambda + 9 = \lambda^2 - (3k_1 - 2k_2 + 14)\lambda + (6k_1 - 15k_2 + 24) = 0
\]

Coefficient matching:

\[
-10 = 3k_1 - 2k_2 + 14 \quad (3)
\]

\[
9 = 24 + 6k_1 - 15k_2 \quad (4)
\]

From Equation (3):

\[
3k_1 = -24 + 2k_2
\]

Plugging into Equation (4):

\[
-48 + 24 + 4k_2 - 15k_2 = 9
\]

\[
k_2 = -3
\]

\[
k_1 = -\frac{24 + 6}{3} = -10
\]