Linearization of systems

One dimensional linear approximation

Consider a differentiable function $f$ of one variable.

$$f : \mathbb{R} \to \mathbb{R}$$

Say we are interested in $f$ in a small, open neighborhood about a particular point $t^*$. $t^*$ is often called the fixed point of the system. Let’s call this open neighborhood $U$. In this case, we can construct a linear approximation of $f$ about the neighborhood $U$. Recall Taylor Approximation,

$$\frac{df}{dt}(t^*) \approx \frac{f(t) - f(t^*)}{t - t^*}$$  for $t \in U$

We can use the above to construct a linear approximation of $f$. Let $f_l$ denote the linear approximation of $f$ about $U$.

$$f_l(t) = \frac{df}{dt}(t^*)(t - t^*) + f(t^*)$$  (1)

Strictly speaking, $f_l$ is an affine approximation unless $t^* = 0$, but the process of obtaining $f_l$ is colloquially called the linearization of $f$.

For example, consider $f(t) = t^2$. We will set the fixed point of the system to be $t^* = 1$. Then,

$$f_l(t) = \frac{df}{dt}(t^*)(t - t^*) + f(t^*)$$
$$= 2(t - 1) + 1$$
$$= 2t - 1$$

Let $\epsilon = 10^{-2}$. Consider the open neighborhood $U = (1 - \epsilon, 1 + \epsilon)$. Let’s plot $f$ and $f_l$ when their respective domains are restricted to $U$. This is seen in Figure ?? on the next page.
Linearization of a system

Consider a continuous, non-linear system with state $\vec{x}(t)$ (which is $n$ dimensional) and input $\bar{u}(t)$ (which is $m$ dimensional) of the form,

$$\frac{d\vec{x}}{dt}(t) = f(\vec{x}(t), \bar{u}(t))$$

To clarify and establish notation, $f$ is a function that takes in $\vec{x}(t)$ and $\bar{u}(t)$ and outputs an $n$ dimensional vector. $f_k(\vec{x}(t), \bar{u}(t))$ refers to the function that is the $k^{th}$ coordinate of the output of $f(\vec{x}(t), \bar{u}(t))$.

Let $\vec{x}^*(t)$ be the desired state trajectory and $\bar{u}^*(t)$ be the desired input. Analogous to (???), we will construct an affine estimate about a neighborhood around the fixed points.

$$f(\vec{x}(t), \bar{u}(t)) \approx f(\vec{x}^*(t), \bar{u}^*(t)) + \nabla_{\vec{x}} f(\vec{x}^*(t), \bar{u}^*(t))(\vec{x}(t) - \vec{x}^*(t)) + \nabla_{\bar{u}} f(\vec{x}^*(t), \bar{u}^*(t))(\bar{u}(t) - \bar{u}^*(t)) \quad (2)$$

Let’s define $\vec{x} = \vec{x} - \vec{x}^*$, $\bar{u} = \bar{u} - \bar{u}^*$. $\nabla f$ is the multidimensional generalization of the derivative, which is constructed as follows.

$$\nabla_{\vec{x}} f(\vec{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}, \bar{u}) & \frac{\partial f_1}{\partial x_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}, \bar{u}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}, \bar{u}) & \frac{\partial f_2}{\partial x_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{x}, \bar{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}, \bar{u}) & \frac{\partial f_n}{\partial x_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{x}, \bar{u}) \end{bmatrix} \quad \text{and} \quad \nabla_{\bar{u}} f(\vec{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\vec{x}, \bar{u}) & \frac{\partial f_1}{\partial u_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_1}{\partial u_m}(\vec{x}, \bar{u}) \\ \frac{\partial f_2}{\partial u_1}(\vec{x}, \bar{u}) & \frac{\partial f_2}{\partial u_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_2}{\partial u_m}(\vec{x}, \bar{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\vec{x}, \bar{u}) & \frac{\partial f_n}{\partial u_2}(\vec{x}, \bar{u}) & \cdots & \frac{\partial f_n}{\partial u_m}(\vec{x}, \bar{u}) \end{bmatrix}$$

Figure 1: **Left:** The linear approximation of the function $f(t) = t^2$ about $t = 1$ is fairly accurate in the range $U = [0.99, 1.01]$. **Right:** Outside of this range, the approximation does not track the function very well.
Note the dimensions of the matrices. It must be notated that, when calculating $\nabla_x f$, $\vec{u}$ is considered constant. Similarly, when calculating $\nabla_u f$, $\vec{x}$ is considered constant.

To continue from (??), note that,

$$\frac{dx}{dt}(t) = f(x(t), u(t)), \quad \frac{dx^*}{dt}(t) = f(x^*(t), u^*(t))$$

and

$$\frac{dx(t)}{dt} = \frac{dx^*(t)}{dt} + \frac{d\vec{x}(t)}{dt}$$

Plugging this back into (??), we get,

$$\frac{d\vec{x}}{dt}(t) \approx \nabla_x f(\vec{x}^*(t), \vec{u}^*(t)) \vec{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t)) \vec{u}(t)$$

Thus, we get linearized version of our system.

$$\frac{d\vec{x}}{dt}(t) \approx \nabla_x f(\vec{x}^*(t), \vec{u}^*(t)) \vec{x}(t) + \nabla_{\vec{u}} f(\vec{x}^*(t), \vec{u}^*(t)) \vec{u}(t)$$

(3)

Note that, unlike the one dimensional linearization example, we are linearizing with respect to $\vec{x}$ and $\vec{u}$. Also, observe that our state variables are now the pertubations $\vec{x}(t)$ and $\vec{u}(t)$.

Questions

1. **Jacobian Warm-Up**

   Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

   $$f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2 \\ x_1 \end{pmatrix}$$

   Calculate its Jacobian.

2. **Linearization**

   Consider a mass attached to two springs:

   ![Diagram of mass attached to two springs](image)

   We assume that each spring is linear with spring constant $k$ and resting length $X_0$. We want to build a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The differential equation modeling this system is

   $$\frac{d^2 y}{dt^2} = -\frac{2k}{m} \left(y - X_0 \frac{y}{\sqrt{y^2 + a^2}}\right).$$

   (a) Write this model in state space form $\dot{x} = f(x)$.

   (b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

   (c) Linearize your model about the equilibrium.