EE16B
Designing Information Devices and Systems II

Lecture 6A
Stability of Linear State Models
Last Time

• Described linearization about an equilibrium point using Taylor approximation
  – Continuous time
  – Discrete time

• Started: Conditions for stability of linear systems
Stability of Linear State Models

Start with scalar system 1st order system:

\[ x(t + 1) = ax(t) + bu(t) \]

Given initial condition \( x(0) \):

\[ x(1) = ax(0) + bu(0) \]

\[ x(2) = ax(1) + bu(1) = a^2x(0) + abu(0) + bu(1) \]

\[ x(3) = a^3x(0) + a^2bu(0) + abu(1) + bu(2) \]

\[ x(t) = a^tx(0) + a^{t-1}bu(0) + a^{t-2}bu(1) + \cdots + a^0bu(t-1) \]
Stability of Linear State Models

Start with scalar system:

\[ x(t + 1) = ax(t) + bu(t) \]

Given initial condition \( x(0) \):

\[
\begin{align*}
x(t) &= a^t x(0) + a^{t-1} bu(0) + a^{t-2} bu(1) + \cdots + a^{0} bu(t - 1) \\
\end{align*}
\]

\[
\begin{align*}
x(t) &= a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \\
\end{align*}
\]
Stability - Definition

• A system is **stable** if $\vec{x}(t)$ is **bounded** for any initial condition $\vec{x}(0)$ and any bounded input sequence $u(0), u(1), \ldots$

• A system is unstable if there is an $\vec{x}(0)$ or a bounded input sequence for which

$$|\vec{x}(t)| \to \infty \quad \text{as} \quad t \to \infty$$
Example

Q) Is this system stable?

\[ x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \]

A) Depends on \( |a| \)


Stability Proof

\[ x(t) = a^t x(0) + \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \]

Claim 1: if \(|a| < 1\) then the system is stable

Proof: \(a^t \to 0\) as \(t \to \infty\) \textit{because} \(|a| < 1\) so,

initial condition always bounded

Sequence is bounded – there exists \(M\) s.t. \(|u(t)| \leq M \ \forall t\)

\[
\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} b u(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b||u(k)| \leq M
\]

\(|a_1| + |a_2| \ ? |a_1 + a_2|\)
Stability Proof Cont.

\[
\left| \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \right| \leq \sum_{k=0}^{t-1} |a^{t-k-1} bu(k)| = \sum_{k=0}^{t-1} |a|^{t-k-1} |b| |u(k)| \leq M
\]

Define: \( s = t - k - 1 \)

\[
\leq \sum_{s=0}^{t-1} |a|^s |b| M = |b| M \sum_{s=0}^{t-1} |a|^s \leq |b| M \frac{1}{1 - |a|}
\]

\[
\sum_{s=0}^{\infty} |a|^s = \frac{1}{1 - |a|}, \quad |a| < 1
\]
Stability Proof Cont.

Claim 2: unstable when $|a| > 1$

Proof: if $x(0) \neq 0$ (even $u(t) = 0 \ \forall \ t$)

$$x(t) = a^t x(0) \rightarrow \infty$$

Q: What if $|a| = 1$, i.e., $a = 1$ or $a = -1$

A: Without input:

$$x(t) = x(0), \quad \text{or} \quad x(t) = (-1)^t x(0)$$

With input $u(t) = M$, $a = 1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} b u(k) \right| = \left| \sum_{k=0}^{t-1} b M \right| \rightarrow \infty \quad \text{Not stable!}$$
Quiz

With input $u(t) = M, a = -1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} u(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq b M$$

Q: what $|u(t)| \leq M$ will make it unstable?
Quiz

With input $u(t) = M$, $a = -1$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b M \right| \leq b M$$

Q: what $|u(t)| \leq M$ will make it unstable?

A: $u(t) = (-1)^t M$

$$\left| \sum_{k=0}^{t-1} a^{t-k-1} bu(k) \right| = \left| \sum_{k=0}^{t-1} (-1)^k b(-1)^k M \right| = \left| \sum_{k=0}^{t-1} b M \right| \to \infty$$
Stability of Linear State Models

Previously, the scalar case:

\[ x(t + 1) = ax(t) + bu(t) \]

\[ |a| < 1 \Rightarrow \text{stable} \]

\[ |a| \geq 1 \Rightarrow \text{unstable} \]

Vector case:

\[ \vec{x}(t + 1) = A\vec{x}(t) + Bu(t) \]

Solve with recursion:

\[ \vec{x}(1) = A\vec{x}(0) + Bu(0) \]

\[ \vec{x}(2) = A^2\vec{x}(0) + A Bu(0) + Bu(1) \]

\[ \vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} Bu(k) \]
Stability – The Vector Case

\[ \mathbf{x}(t) = A^t \mathbf{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} B u(k) \]

Q: How do we determine stability?

A (partial): Not simple as in the scalar case – the state variables and inputs are coupled.

Approach: Let’s change variables, to decouple them
Change of Variables - Diagonalization

\[
\vec{x}(t + 1) = A\vec{x}(t) + Bu(t)
\]
\[
\vec{z}(t) = T\vec{x}(t)
\]
\[
\vec{z}(t + 1) = T\vec{x}(t + 1)
\]
\[
= TA\vec{x}(t) + TBu(t)
\]

\[
A_{\text{new}} = TAT^{-1}
\]
\[
B_{\text{new}} = TB
\]

Q: What \(T\) to choose?

A: Choose \(T\) s.t. \(A_{\text{new}}\) is diagonal

Similarity transformation

Remember eigen values of new system same as original!
Diagonalization

\[ A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \]

\[ \ddot{z}(t + 1) = A_{\text{new}} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + B_{\text{new}} u(t) \]

\[ z_1(t + 1) = \lambda_1 z_1(t) + v_1(t) \]
Diagonalization = decoupling!

\[
\begin{align*}
    z_1(t + 1) &= \lambda_1 z_1(t) + v_1(t) \\
    z_2(t + 1) &= \lambda_2 z_2(t) + v_2(t) \\
    &\vdots \\
    z_n(t + 1) &= \lambda_n z_n(t) + v_n(t)
\end{align*}
\]

Stable if: \( |\lambda_i| < 1, \quad i = 1, 2, \cdots, n \)

unstable if: \( |\lambda_i| \geq 1, \quad i = 1, 2, \cdots, n \)

Remember eigen values of new system same as original!
Stability Cont.

What if $a$ is complex valued?

- Stable if: $|\lambda_i| < 1$, $i = 1, 2, \ldots, n$
- Unstable if: $|\lambda_i| \geq 1$, $i = 1, 2, \ldots, n$
Non-Diagonalizable Systems

Q: What if $A$ is not diagonalizable?

A: Transform to upper diagonal form (always possible)

$$T A T^{-1} = \begin{bmatrix}
\lambda_1 & \ast & \cdots & \ast \\
0 & \lambda_2 & \cdots & \ast \\
& & \ddots & \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}$$

beyond 16B material

Stable if: $|\lambda_i| < 1$, $i = 1, 2, \ldots, n$

unstable if: $|\lambda_i| \geq 1$, $i = 1, 2, \ldots, n$
Non-Diagonalizable Proof

Show stability for $z_n$:

$$z_n(t + 1) = \lambda_n z_n(t) + v_n(t)$$

$z_n$ is bounded, show stability for $z_{n-1}$:

$$z_{n-1}(t + 1) = \lambda_{n-1} z_{n-1}(t) + \lambda z_n(t) + v_{n-1}(t)$$

Bounded if $|\lambda_{n-1}| < 1$

show stability for $z_i$ recursively!

Example of non-diagonalizable:

$$\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}$$
Stability of Cont.-Time Linear Systems

\[
\frac{d}{dt} \bar{x}(t) = A\bar{x}(t) + Bu(t)
\]

Start with scalar \(x(t)\):

\[
\frac{d}{dt} x(t) = ax(t) + bu(t)
\]

\[
x(t) = e^{at}x(0) + b \int_0^t e^{a(t-s)} u(s) ds
\]

Initial condition

Due to input
Stability of Cont.-Time Linear Systems

\[ \frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + B u(t) \]

Start with scalar \( x(t) \):

\[ \frac{d}{dt} x(t) = ax(t) + bu(t) \]

\[ x(t) = e^{at} x(0) + b \int_0^t e^{a(t-s)} u(s) ds \]

Initial condition \quad Due to input

Q: When is the system stable?
Stability of Cont.-Time Linear Systems

\[
\frac{dx(t)}{dt} = ax(t) + bu(t)
\]

\[
x(t) = e^{at}x(0) + b \int_0^t e^{a(t-s)} u(s) \, ds
\]

Initial condition

Due to input

Q: When is the system stable?
A: For \( a < 0 \)

Proof outline:
Show: \( e^{at} \to 0, \quad t \to \infty \)
if \( |u(s)| \leq M \forall s \Rightarrow \int \{ \} < \text{Const} \)
Stability of Cont.-Time Linear Systems

$$x(t) = e^{at}x(0) + b \int_0^t e^{a(t-s)}u(s)ds$$

Q: When is the system unstable?

A: For $a \geq 0$

Proof: choose $x(0) \neq 0$ and $u(t) = M$

either “due to input” or “due to initial condition” explodes
Stability of Cont.-Time Linear Systems

Summary:

\[ a < 0 \Rightarrow \text{stable} \]
\[ a \geq 0 \Rightarrow \text{unstable} \]

If \( a \) is complex, then:

\[ \text{Re}\{a\} < 0 \Rightarrow \text{stable} \]
\[ \text{Re}\{a\} \geq 0 \Rightarrow \text{unstable} \]

\[ |e^{a_r + i a_i}| = |e^{a_r}| \cdot |e^{ia_i}| = |e^{a_r}| \]
Stability of Cont.-Time Linear Systems

Vector Case:

\[
\frac{d}{dt} \vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)
\]

Diagonalize:

\[
A_{\text{new}} = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
& & \ddots & \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
\]

\[
\frac{d}{dt} z_i(t) = \lambda_i z_i(t) + v_i(t)
\]

\[
\vec{z}(t) = T\vec{x}(t)
\]
Stability of Cont.-Time Linear Systems

Stability test for

\[ \frac{d}{dt} \bar{x}(t) = A\bar{x}(t) + B\bar{u}(t) \]

\[ \text{Re}\{\lambda_i(A)\} < 0 \quad \forall i \mid i = 1, 2, \ldots, n \implies \text{stable} \]

\[ \text{Re}\{\lambda_i(A)\} \geq 0 \quad \exists i \mid i = 1, 2, \ldots, n \implies \text{unstable} \]
Stability -- Summary

Discrete-Time

\[ |\lambda_i(A)| < 1 \]

Continuous-Time

\[ \text{Real}\{\lambda_i(A)\} < 0 \]

Stay away from boundaries! System uncertainty can move you over to unstable region.
Back to the Pendulum

\[ A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \]

\[ A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} \]

\[
|\lambda I - A_{\text{down}}| = \begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m} \lambda + \frac{g}{l} = 0
\]

\[
\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4 \frac{g}{l}}
\]
Back to the Pendulum

\[ \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}} \]

If \( \frac{k^2}{m^2} \geq 4\frac{g}{l} \), i.e, sqrt is real, then \( \frac{k}{2m} \geq \frac{1}{2} \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}} \)

So, \( \lambda_{1,2} \) always negative -- stable!

If \( \frac{k^2}{m^2} < 4\frac{g}{l} \), i.e, sqrt is imaginary, then \( \text{Re}\{\lambda_{1,2}\} = -\frac{k}{2m} \)

So, \( \text{Re}\{\lambda_{1,2}\} \) always negative -- stable!
Back to the Pendulum

\[
A_{\text{down}} = \begin{bmatrix}
0 & 1 \\
-\frac{g}{l} & -\frac{k}{m}
\end{bmatrix}
\]

\[
A_{\text{up}} = \begin{bmatrix}
0 & 1 \\
g/l & -\frac{k}{m}
\end{bmatrix}
\]

\[|\lambda I - A_{\text{up}}| = \begin{bmatrix}
\lambda & -1 \\
-\frac{g}{l} & \lambda + \frac{k}{m}
\end{bmatrix} = \lambda^2 + \frac{k}{m} \lambda - \frac{g}{l} = 0\]

\[\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} + 4 \frac{g}{l}}\]

\[\lambda_1 > 0\]

\[\lambda_2 < 0\]
Back to the Pendulum

\[ \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2} \sqrt{\frac{k^2}{m^2} + 4 \frac{g}{l}} \]

\[ \lambda_1 > 0 \]
\[ \lambda_2 < 0 \]

long \ l \short \ l
Predicting System Behavior

Discrete Time

\[ \lambda^t \]

\[ z(t + 1) = \lambda_i z(t) \]

Soln: \[ \lambda_i^t z(0) \]
• If $\lambda$ is complex

$$\lambda^t = (|\lambda| e^{j\omega})^t$$

$$= |\lambda|^t e^{j\omega t}$$

$$\lambda = |\lambda| e^{j\omega} \quad \text{(Euler)}$$
• If $\lambda$ is complex

$$\lambda^t = (|\lambda|e^{j\omega})^t$$

$$= |\lambda|^t e^{j\omega t}$$

• Continuous time:

$$\frac{d}{dt}Z_i(t) = \lambda_i Z_i(t) \Rightarrow e^{\lambda_i t} Z_i(0)$$

Q) What does $e^{\lambda t}$ look like for different choices of $\lambda$?

A) $\lambda = v + j\omega$ \Rightarrow $e^{\lambda t} = e^{vt} e^{j\omega t}$
Summary

• Derived stability conditions for vector discrete and continuous systems

• Showed that it is easy to analyze with change of variables!

• Prediction of system behaviour for different eigenvalues
  – For discrete – Phase (angle) determines frequency and magnitude determines relaxation
  – For continuous – Real part = relaxation, imaginary = frequency of oscillations

• Next time: Control design – putting the eigenvalues where we want them!