1. Complex Algebra

(a) Express the following values in polar forms: $-1, j, -j, \sqrt{j},$ and $\sqrt{-j}$. Recall $j = \sqrt{-1}$.

(b) Represent $\sin \theta$ and $\cos \theta$ using complex exponentials. (*Hint:* Use Euler’s identity $e^{j\theta} = \cos \theta + j\sin \theta$.)

(c) For complex number $z = x + jy$ show that $|z| = \sqrt{\overline{z}z}$, where $\overline{z}$ is the complex conjugate of $z$.

For the next two parts, consider two complex numbers $A = 1 - j\sqrt{3}$ and $B = \sqrt{3} + j$.

(d) Express $A$ and $B$ in polar form.

(e) Find $AB$, $A\overline{B}$, $A\overline{B}$, and $\sqrt{B}$.

2. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations

$$\frac{d}{dt}x_1(t) = \lambda x_1(t) \quad (1)$$
$$\frac{d}{dt}x_2(t) = \overline{\lambda} x_2(t) \quad (2)$$

with initial conditions $x_1(0) = c_0$ and $x_2(0) = \tau_0$, where $\lambda$ and $c_0$ are complex numbers and $\overline{\lambda}$ and $\tau_0$ are their complex conjugates, respectively.

Suppose now that we have the following different variables related to the original ones:

$$y_1(t) = ax_1(t) + \overline{a}x_2(t) \quad (3)$$
$$y_2(t) = bx_1(t) + \overline{b}x_2(t) \quad (4)$$

where $a$ and $b$ are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written:

$$a = a_r + ja_i,$$
$$\overline{a} = a_r - ja_i,$$
$$b = b_r + jb_i,$$
$$\overline{b} = b_r - jb_i,$$

where $a_r, a_i, b_r, b_i$ are all real numbers.

(a) First, assume that $\lambda = j = \sqrt{-1}$ in the equations for $x_1(t)$ and $x_2(t)$ above. Solve $x_1(t)$ and $x_2(t)$.

(b) How do the initial conditions for $\overline{x}(t)$ translate into the initial conditions for $\overline{y}(t)$?

(c) Write out a system of differential equations using $\frac{d}{dt}y_1(t)$ and $y_1(t)$.
(d) Suppose we know \( x_1(t) \) and \( x_2(t) \) are complex conjugates of each other. What will this say about \( y_1(t) \) and \( y_2(t) \)?

(e) Find the eigenvalues \( \lambda_1, \lambda_2 \) and associated eigenspaces for the differential equation matrix for \( \vec{y}(t) \) above.

(f) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables \( z_{\lambda_1}(t), z_{\lambda_2}(t) \). (These variables should be in eigenbasis-aligned coordinates.)

(g) Solve the differential equation for \( z_{\lambda}(t) \) in the eigenbasis.

(h) Convert your solution back into the \( \vec{y}(t) \) coordinates to find \( \vec{y}(t) \).

(i) Repeat the above for general complex \( \lambda \).

3. RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:

![RLC Circuit Diagram]

Assume the circuit above has reached steady state for \( t < 0 \). At time \( t = 0 \), the switch changes state and disconnects the voltage source, replacing it with a short.

(a) Write the system of differential equations in terms of state variables \( x_1(t) = I_L(t) \) and \( x_2(t) = V_C(t) \) that describes this circuit for \( t \geq 0 \). Leave the system symbolic in terms of \( V_s, L, R, \) and \( C \).

(b) Write the system of equations in vector/matrix form with the vector state variable \( \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \).

This should be in the form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \) with a \( 2 \times 2 \) matrix \( A \).

(c) Find the eigenvalues of the \( A \) matrix symbolically. (Hint: the quadratic formula will be involved.)

(d) Under what condition on the circuit parameters \( R, L, C \) are there going to be a pair of distinct real eigenvalues of \( A \)?

(e) Under what condition on the circuit parameters \( R, L, C \) are there going to be a pair of purely imaginary eigenvalues of \( A \)?

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues \( \lambda_1, \lambda_2 \) so that \( \lambda_1 \neq \lambda_2 \), find eigenvectors \( \vec{v}_{\lambda_1}, \vec{v}_{\lambda_2} \) corresponding to them.

(HINT: Rather than trying to find the relevant nullspaces, etc., you might just want to try to find eigenvectors of the form \( \begin{bmatrix} 1 \\ ? \end{bmatrix} \) where we just want to find the missing entry. Can you see from the structure of the \( A \) matrix why we might want to try that guess?)
(g) Assuming circuit parameters such that the two eigenvalues of $A$ are distinct, let $V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\tilde{x}}(t)$. **What is the $\tilde{A}$ so that $\frac{d}{dt}\vec{\tilde{x}}(t) = \tilde{A}\vec{\tilde{x}}(t)$?** It is fine to have your answer expressed symbolically using $\lambda_1, \lambda_2$. 

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