1 Complex Numbers

A complex number $z$ is an ordered pair $(x,y)$, where $x$ and $y$ are real numbers, written as $z = x + jy$ where $j = \sqrt{-1}$. A complex number can also be written in polar form as follows:

$$z = |z|e^{j\theta}$$

The magnitude $z$ is denoted as $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

The phase or argument of a complex number is denoted as $\theta$ and is given by

$$\theta = \text{atan2}(y,x).$$

Here, atan2($y,x$) is a function that returns the angle from the positive x-axis to the vector from the origin to the point $(x,y)$.

The complex conjugate of a complex number $z$ is denoted by $\bar{z}$ (or might also be written $z^*$) and is given by

$$\bar{z} = x - jy.$$

Euler’s Identity is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

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$^1$See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at [https://en.wikipedia.org/wiki/Atan2](https://en.wikipedia.org/wiki/Atan2).
With this definition, the polar representation of a complex number will make more sense. Note that
\[ |z| e^{j\theta} = |z| \cos(\theta) + j |z| \sin(\theta). \]

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case \( |z| \) is the magnitude and \( e^{j\theta} \) is the unit vector that defines the direction.

2 Useful Identities

**Complex Number Properties**

**Rectangular vs. polar forms:** \( z = x + jy = |z| e^{j\theta} \)
where \( |z| = \sqrt{x^2 + y^2}, \ \theta = \arctan(y/x) \). We can also write \( x = |z| \cos \theta, \ y = |z| \sin \theta \).

**Euler’s identity:** \( e^{j\theta} = \cos \theta + j \sin \theta \)
\[ \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \ \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \]

**Complex conjugate:** \( \bar{z} = x - jy = |z| e^{-j\theta} \)
\[ (z + w) = \bar{z} + \bar{w}, \ (\bar{z} - w) = z - w \]
\[ (\bar{zw}) = \bar{z} \bar{w}, \ (\bar{z}/w) = \bar{z}/\bar{w} \]
\( \bar{z} = z \iff z \) is real
\[ (\bar{z})^n = (z)^n \]

**Complex Algebra**

Let \( z_1 = x_1 + jy_1 = |z_1| e^{j\theta_1}, z_2 = x_2 + jy_2 = |z_2| e^{j\theta_2} \).
(Note that we adopt the easier representation between rectangular form and polar form.)

**Addition:** \( z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \)

**Multiplication:** \( z_1 z_2 = |z_1||z_2| e^{j(\theta_1 + \theta_2)} \)

**Division:** \( \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{j(\theta_1 - \theta_2)} \)

**Power:** \( z_1^n = |z_1|^n e^{jn \theta_1} \)
\[ \frac{1}{z_1} = \pm |z_1| \frac{1}{|z_1|} e^{-j\frac{\theta_1}{2}} \]

**Useful Relations**
\[ -1 = j^2 = e^{j\pi} = e^{-j\pi} \]
\[ j = e^{j\frac{\pi}{2}} = \sqrt{-1} \]
\[ -j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \]
\[ \sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1 + j)}{\sqrt{2}} \]
\[ \sqrt{-j} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1 - j)}{\sqrt{2}} \]

3 Phasors

We consider sinusoidal voltages and currents of a specific form:

| Voltage | \( v(t) = V_0 \cos(\omega t + \phi_v) \) |
| Current | \( i(t) = I_0 \cos(\omega t + \phi_i) \) |

where,

(a) \( V_0 \) is the voltage amplitude and is the highest value of voltage \( v(t) \) will attain at any time. Similarly, \( I_0 \) is the current amplitude.

(b) \( \omega \) is the angular frequency of oscillation. \( \omega \) is related to frequency by \( \omega = 2\pi f \). Frequency \( f \) is the number of oscillation cycles that occur per second. If \( T \) is the period of the sinusoid (that is, the amount of time it takes for one complete cycle to occur) the frequency is \( f = \frac{1}{T} \).
(c) $\phi_v$ and $\phi_i$ are the phase terms of the voltage and current respectively. These capture a delay, or a shift in time, of the sinusoid.

We know from Euler’s identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = \frac{V_0}{2} e^{j\omega t + j\phi_v} + \frac{V_0}{2} e^{-j\omega t - j\phi_v} = \frac{V_0}{2} e^{j\phi_v} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi_v} e^{-j\omega t}$$

The coefficient of the $e^{j\omega t}$ term in Equation 1 is called the phasor form of this signal:

$$\vec{V} = \frac{1}{2} V_0 e^{j\phi_v}$$

The complex conjugate of the phasor is the coefficient of the $e^{-j\omega t}$ term in Equation 1 and is given by:

$$\overline{\vec{V}} = \frac{1}{2} V_0 e^{-j\phi_v}$$

The phasor representation is a constant that contains the magnitude and phase information of the signal. The time-varying part of the signal does not need to be explicitly represented, because it is given by $e^{j\omega t}$, which is always implicit when using phasors. Phasors let us handle sinusoidal signals much more easily. They are powerful because they let us use DC-like (16A style) circuit analysis techniques, which we already know, to analyze circuits with sinusoidal voltages and currents.

**Note:** We can only use phasors if we know that all of our signals are sinusoids. We will see later in 16B (Module 4) why this is not too restrictive a condition.

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\vec{V}$ is the phasor.

$$V_0 \cos(\omega t + \phi_v) = \vec{V} e^{j\omega t} + \overline{\vec{V}} e^{-j\omega t}$$

The standard forms for voltage and current phasors are given below:

| Voltage | $\vec{V} = \frac{1}{2} V_0 e^{j\phi_v}$ |
| Current | $\vec{I} = \frac{1}{2} I_0 e^{j\phi_i}$ |

We define the impedance of a circuit component to be $Z = \frac{\vec{V}}{\vec{I}}$, where $\vec{V}$ and $\vec{I}$ represent the voltage across and the current through the component, respectively. For capacitors and inductors, this impedance will generally depend on $j\omega$.
3.1 Phasor Relationship for Resistors

Consider a simple resistor circuit as in Figure 2 with current being

\[ i(t) = I_0 \cos(\omega t + \phi) = \frac{I_0}{2} e^{j\phi} e^{j\omega t} + \frac{I_0}{2} e^{-j\phi} e^{-j\omega t} \]

By Ohm’s law,

\[ v(t) = i(t)R \]
\[ = I_0R \cos(\omega t + \phi) = \frac{I_0R}{2} e^{j\phi} e^{j\omega t} + \frac{I_0R}{2} e^{-j\phi} e^{-j\omega t} \]

In the phasor domain, for both \( s = +j\omega \) and \( s = -j\omega \)

\[ \tilde{V} = R\tilde{I} \]

We usually refer to the impedance of the resistor, \( Z_R \), in the phasor domain. Since \( Z_R = R \), we can also write:

\[ \tilde{V} = Z_R\tilde{I} \]

3.2 Phasor Relationship for Capacitors

Consider a capacitor circuit as in Figure 3 with voltage being

\[ v(t) = V_0 \cos(\omega t + \phi) \]
By the capacitor equation, and expanding \( \cos \) in the complex exponential domain:

\[
v(t) = V_0 \cos(\omega t + \phi) = \frac{V_0}{2} e^{j(\omega t + \phi)} + \frac{V_0}{2} e^{-j(\omega t + \phi)}
\]

\[
= \frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t}
\]

\[
= \bar{V} e^{j\omega t} + \bar{V} e^{-j\omega t}
\]

\[
i(t) = C \frac{d}{dt} v(t)
\]

\[
= C \frac{d}{dt} \left( \frac{V_0}{2} e^{j\phi} e^{j\omega t} + \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \right)
\]

\[
= j\omega C \left( \frac{V_0}{2} e^{j\phi} e^{j\omega t} - j\omega C \frac{V_0}{2} e^{-j\phi} e^{-j\omega t} \right)
\]

Considering the \( s = +j\omega \) term:

\[
\tilde{I} = j\omega C \frac{V_0}{2} e^{j\phi} = j\omega C \bar{V}
\]

This can also be derived in the sinusoid domain:

\[
i(t) = C \frac{dv}{dt}(t) = -CV_0 \omega \sin(\omega t + \phi)
\]

\[
= -CV_0 \omega \left( -\cos(\omega t + \phi + \frac{\pi}{2}) \right)
\]

\[
= CV_0 \omega \cos(\omega t + \phi + \frac{\pi}{2})
\]

\[
= (\omega C) V_0 \cos(\omega t + \phi + \frac{\pi}{2})
\]

In the phasor domain,

\[
\tilde{I} = \omega C e^{j\frac{\pi}{2}} \bar{V} = j\omega C \bar{V}
\]

The impedance of a capacitor is an abstraction to model the capacitor in a manner similar to a resistor in the phasor domain. This is denoted \( Z_C \), which is given by:

\[
Z_C = \frac{\bar{V}}{\tilde{I}} = \frac{1}{j\omega C} = \frac{1}{sC}
\]

for \( s = j\omega \).

1. Cosine vs Sine Relation

(a) What value of \( \phi \) makes the following relation true?

\[
\sin(\omega t) = \cos(\omega t + \phi)
\]
2. Inductor Impedance
Given the voltage-current relationship of an inductor $V = L \frac{di}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

3. Complex Matrix Inverse
Consider a complex matrix

$$M = M_r + jM_i$$

and its inverse

$$N = N_r + jN_i$$

(a) Show that the inverse of $\overline{M} = M_r - jM_i$ (the complex conjugate of $M$) is equal to $\overline{N} = N_r - jN_i$ (the complex conjugate of $N$).

4. Phasor Analysis
Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \tilde{X} e^{j\omega t} + \tilde{X} e^{-j\omega t},$$

where $\tilde{X}$ is a time-independent function called the **phasor** representation of $x(t)$ (recall that $\overline{a}$ denotes the complex conjugate of $a$). Note that 1) $\tilde{X}$ and $\overline{\tilde{X}}$ are complex conjugates of each other, 2) $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates of each other, and 3) that $\tilde{X} e^{j\omega t}$ and $\overline{\tilde{X}} e^{-j\omega t}$ are also complex conjugates of each other.

The phasor analysis method consists of five steps. Consider the RC circuit below.

![RC Circuit Diagram]

The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right),$$

with $\omega = 1 \times 10^3 \text{ rad/s}$, $R = \sqrt{3} k\Omega$, and $C = 1 \mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

(a) **Step 1: Write sources as exponentials:** $\tilde{A} e^{j\omega t} + \overline{\tilde{A}} e^{-j\omega t}$

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. Convert $v_s(t)$ into a exponential and write down its phasor representation $\tilde{V}_s$. 

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(b) **Step 2: Transform circuits to phasor domain for** \( s = +j\omega \)

The voltage source is represented by its phasor \( \tilde{V}_s \). The current \( i(t) \) is related to its phasor counterpart \( \tilde{I} \) by

\[
i(t) = \tilde{I} e^{j\omega t} + \tilde{I} e^{-j\omega t}.
\]

What are the impedances of the resistor, \( Z_R \), and capacitor, \( Z_C \)? We sometimes also refer to this as the "phasor representation" of \( R \) and \( C \).

(c) **Step 3: Cast the branch and element equations in phasor domain**

Use Kirchhoff’s laws to write down a loop equation that relates all phasors in Step 2.

(d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for \( \tilde{I} \) and \( \tilde{V}_C \). What is the polar form of \( \tilde{I} \) and \( \tilde{V}_C \)? Polar form is given by \( Ae^{j\theta} \), where \( A \) is a positive real number.

(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is \( i(t) \) and \( v_C(t) \)? What is the phase difference between \( i(t) \) and \( v_C(t) \)?

### 5. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

\[
v_s(t) = B \cos(\omega t - \phi)
\]

![RLC Circuit Diagram](image)

(a) Write out the phasor representations of \( v_s(t) \), \( R \), \( C \), and \( L \).

(b) Use Kirchhoff’s laws to write down a loop equation relating the phasors in the previous part.

(c) Solve the equation in the previous step for the current \( \tilde{I} \). What is the magnitude and phase of the polar form of \( \tilde{I} \)?

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