1. RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:

Assume the circuit above has reached steady state for \( t < 0 \). At time \( t = 0 \), the switch changes state and disconnects the voltage source, replacing it with a short.

In this problem, the current through the inductor and the voltage across the capacitor are the natural physical state variables since these are what correlate to how energy is actually stored in the system. (A magnetic field through the inductor and an electric field within the capacitor.)

(a) Write the system of differential equations in terms of state variables \( x_1(t) = I_L(t) \) and \( x_2(t) = V_C(t) \) that describes this circuit for \( t \geq 0 \). Leave the system symbolic in terms of \( V_s, L, R, \) and \( C \).

Solution: For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let’s consider the capacitor equation \( I_C(t) = C \frac{d}{dt} V_C(t) \). In this circuit, \( I_C(t) = I_L(t) \), so we can write

\[
I_C(t) = C \frac{d}{dt} V_C(t) = I_L(t) \tag{1}
\]

\[
\frac{d}{dt} V_C(t) = \frac{1}{C} I_L(t). \tag{2}
\]

If we use the state variable names, we can write this as

\[
\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t), \tag{3}
\]

so now we have one differential equation.
For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have
\[ V_C(t) + V_R(t) + V_L(t) = 0. \] (4)

Using Ohm’s Law and the inductor equation \( V_L = L \frac{d}{dt} I_L(t) \), we can write this as
\[ V_C(t) + R I_L(t) + L \frac{d}{dt} I_L(t) = 0, \] (5)

which we can rewrite as
\[ \frac{d}{dt} I_L(t) = -\frac{R}{L} I_L(t) - \frac{1}{L} V_C(t). \] (6)

If we use the state variable names, this becomes
\[ \frac{d}{dt} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t), \] (7)

and we have a second differential equation.

To summarize the final system is
\[ \frac{d}{dt} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t) \] (8)
\[ \frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t). \] (9)

(b) **Write the system of equations in vector/matrix form with the vector state variable** \( \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \).

This should be in the form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \) with a \( 2 \times 2 \) matrix \( A \).

**Solution:** By inspection from the previous part, we have
\[ \begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \] (10)

which is in the form \( \frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \), with
\[ A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \] (11)

(c) **Find the eigenvalues of the** \( A \) **matrix symbolically.**

*(Hint: the quadratic formula will be involved.)*

**Solution:** To find the eigenvalues, we’ll solve \( \det(A - \lambda I) = 0 \). In other words, we want to find \( \lambda \) such that
\[ \det(A - \lambda I) = \det \begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix} \] (12)
\[ = -\lambda \begin{bmatrix} -\frac{R}{L} - \lambda \end{bmatrix} + \frac{1}{LC} \] (13)
\[ = \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} = 0. \] (14)
The Quadratic Formula gives
\[ \lambda = -\frac{1}{2} R \pm \frac{1}{2} \sqrt{(\frac{R}{L})^2 - \frac{4}{LC}}. \] (15)

(d) Under what condition on the circuit parameters \( R, L, C \) are there going to be a pair of distinct purely real eigenvalues of \( A \)?

**Solution:** For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need
\[ \frac{R^2}{L^2} - \frac{4}{LC} > 0, \] (16)
or, equivalently,
\[ R > 2\sqrt{\frac{L}{C}}. \] (17)

(e) Under what condition on the circuit parameters \( R, L, C \) are there going to be a pair of purely imaginary eigenvalues of \( A \)?

**Solution:** The only way for both eigenvalues to be purely imaginary is to have \( R = 0 \). In this case, the eigenvalues would be
\[ \lambda = \pm j\sqrt{\frac{1}{LC}}. \] (18)

(f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues \( \lambda_1, \lambda_2 \) so that \( \lambda_1 \neq \lambda_2 \), find eigenvectors \( \vec{v}_{\lambda_1}, \vec{v}_{\lambda_2} \) corresponding to them.

(HINT: Rather than trying to find the relevant nullspaces, etc., you might just want to try to find eigenvectors of the form \( \begin{bmatrix} 1 \\ ? \end{bmatrix} \) where we just want to find the missing entry. Can you see from the structure of the \( A \) matrix why we might want to try that guess?)

**Solution:**

The easy way is just to remember what an eigenvector is. We want \( A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i} \). So, we can try to follow the hint:

\[ \begin{bmatrix} -\frac{R}{L} & -\frac{1}{C} \\ \frac{1}{C} & -\frac{L}{C} \end{bmatrix} \begin{bmatrix} 1 \\ ? \end{bmatrix} = \begin{bmatrix} \lambda_i \\ \lambda_i \end{bmatrix} \] (19)

We want the second entry to be \(?\lambda_i\), but instead we have \( \frac{1}{C} \), and so the \(? = \frac{1}{C\lambda_i} \). This immediately gives us
\[ \vec{v}_{\lambda_1} = \begin{bmatrix} \frac{1}{\lambda_1 C} \\ \frac{1}{\lambda_1 C} \end{bmatrix} \]
\[ \vec{v}_{\lambda_2} = \begin{bmatrix} \frac{1}{\lambda_2 C} \\ \frac{1}{\lambda_2 C} \end{bmatrix} \]

Since these by construction obey \( A\vec{v}_{\lambda_i} = \lambda_i \vec{v}_{\lambda_i} \) in the second position and we know that we have the flexibility to scale eigenvectors, they must be eigenvectors. i.e. We always have the flexibility to scale
a single nonzero position to be 1. Putting a zero in the first position does not result in an eigenvector by inspection of the first row since we would get \(-\frac{R}{L}\) unless \(? = 0\) and \(\vec{0}\) is never a valid eigenvector. (This is how being able to argue rigorously lets you avoid some algebra.) Of course, you can grind out the minor algebra to just verify from the expressions for the eigenvalues in (15) that they indeed do satisfy \(AV\lambda_i = \lambda_i V\lambda_i\) in the first position as well since \(-\frac{R}{L} - \frac{1}{\lambda_i C}\) indeed equals \(\lambda_i\) for the solutions given in (15). Alternatively, you can try to use the standard approach of finding the nullspace of \(A - \lambda I\). Since the matrix \(A\) has a zero in the second row, and we know

\[
\begin{bmatrix}
-\frac{R}{L} - \frac{1}{\lambda_1 C} & -\frac{1}{\lambda_1} \\
\frac{1}{\lambda_1} & -\frac{1}{\lambda_1}
\end{bmatrix}
\vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

and

\[
\begin{bmatrix}
-\frac{R}{L} - \frac{1}{\lambda_2 C} & -\frac{1}{\lambda_2} \\
\frac{1}{\lambda_2} & -\frac{1}{\lambda_2}
\end{bmatrix}
\vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

then it is clear that the nullspace can be found as anything along the following two vectors:

\[
\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}
\]

\[
\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}
\]

These are clearly linearly independent since \(\lambda_1 \neq \lambda_2\).

Either way works.

(g) Assuming circuit parameters such that the two eigenvalues of \(A\) are distinct, let \(V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]\) be a specific eigenbasis. Consider a coordinate system for which we can write \(\vec{x}(t) = V\vec{\tilde{x}}(t)\). What is the \(\tilde{A}\) so that \(\frac{d}{dt}\vec{\tilde{x}}(t) = \tilde{A}\vec{\tilde{x}}(t)\)? It is fine to have your answer expressed symbolically using \(\lambda_1, \lambda_2\).

**Solution:** \(V\) is given by:

\[
V = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}
\]

We know that \(V\) transforms from the \(\vec{x}\) coordinate frame to the \(x\) coordinate frame, \(V^{-1}\) transforms back, and \(A\) takes gives the relationship from \(x\) to \(\frac{d}{dt}\vec{x}\).

Therefore to go from \(\vec{x}\) to \(\frac{d}{dt}\vec{x}\):

\[
\tilde{A} = V^{-1}AV = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}
\begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1}
\begin{bmatrix} -\frac{R}{L} & -\frac{1}{\lambda_1 C} \\ \frac{1}{\lambda_1} & 0 \end{bmatrix}
\begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}
\]

\[
\tilde{A} = V^{-1}AV = \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2}
\begin{bmatrix} \frac{1}{\lambda_1 C} & -1 \\ \frac{1}{\lambda_2 C} & 1 \end{bmatrix}
\begin{bmatrix} -\frac{R}{L} & -\frac{1}{\lambda_2 C} \\ \frac{1}{\lambda_2} & 0 \end{bmatrix}
\begin{bmatrix} \frac{1}{\lambda_1 C} & 1 \\ \frac{1}{\lambda_2 C} & \frac{1}{\lambda_2 C} \end{bmatrix}
\]

\[
\tilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\]
You didn’t have to multiply things out explicitly. You could have just noticed that the $A$ matrix times the $V$ matrix would give columns that were $\lambda_i$ times $\vec{v}_i$ each, and then multiplying that by $V^{-1}$ would just pick out the $\lambda_i$ on the diagonals and zeros on the off-diagonals since $V^{-1}V = I$.

2. RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:

Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

(a) Suppose $R = 1 \text{ k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\tilde{x}(0)$. Recall that $\tilde{x}$ is in the changed “nice” eigenbasis coordinates from the first problem.

Solution: First of all, we must state the initial conditions for $\tilde{x}(0)$. If the circuit is in steady state before $t = 0$, then no current is flowing and the entire voltage drop is across the capacitor. Therefore:

$x_1(0) = I_L(0) = 0$
$x_2(0) = V_C(0) = V_s = 1$

Under these conditions, we can solve for

$\lambda_1 = -1.0 \times 10^5$, $\lambda_2 = -4.0 \times 10^7$

$V^{-1} = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix}$

$\tilde{x}(0) = V^{-1}\tilde{x}(0) = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \end{bmatrix}$

(b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

Note: Because there is a lot of resistance, this is called the “overdamped” case. However, at this particular point in this problem, you probably have no intuition for what is “over” about it. That is fine. There are more problems coming to help us understand this.

Solution: Plugging in for the component values gives:
\[ A = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix} \]

These eigenvalues are the negative reciprocals of the relevant time constants for these modes.

\[ \begin{bmatrix} \frac{d}{dt} \tilde{x}_1(t) \\ \frac{d}{dt} \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \quad (20) \]

Therefore:

\[ \tilde{x}_1(t) = K_1 e^{-1.0 \times 10^5 t} \]
\[ \tilde{x}_2(t) = K_2 e^{-4.0 \times 10^7 t} \]

Solving for \( K \) with the initial condition gives:

\[ \tilde{x}_1(t) = -0.001 e^{-1.0 \times 10^5 t} \]
\[ \tilde{x}_2(t) = 0.001 e^{-4.0 \times 10^7 t} \]

Converting back to the \( \tilde{x} \) coordinates:

\[ \tilde{x}(t) = V \tilde{x}(t) = \begin{bmatrix} 1 & 1 \\ -1000 & -2.5 \end{bmatrix} \tilde{x}(t) \]
\[ x_1(t) = -0.001 e^{-1.0 \times 10^5 t} + 0.001 e^{-4.0 \times 10^7 t} \]
\[ x_2(t) = e^{-1.0 \times 10^5 t} - 0.0025 e^{-4.0 \times 10^7 t} \]

3. **Complex Numbers (PRACTICE)**

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number \( z \) is often represented in Cartesian form.

\[ z = x + jy \text{ with } \text{Re}\{z\} = x \text{ and } \text{Im}\{z\} = y \]

See Figure[1] for a visualization of \( z \) in the complex plane.
In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

(a) **Calculate the length of** $z$ **in terms of** $x$ **and** $y$ **as shown in Figure 1**. This is the magnitude of a complex number and is denoted by $|z|$ or $r$.

(Hint: Use the Pythagorean theorem.)

**Solution:**

$$r = \sqrt{x^2 + y^2} = |z|$$

(b) **Represent** $x$, **the real part of** $z$, **and** $y$, **the imaginary part of** $z$, **in terms of** $r$ **and** $\theta$.

**Solution:**

$$x = r \cos(\theta) \text{ and } y = r \sin(\theta)$$

(c) **Substitute for** $x$ **and** $y$ **in** $z$. Use Euler’s identity $e^{j\theta} = \cos \theta + j \sin \theta$ to conclude that,

**Solution:**

$$z = re^{j\theta}$$

\footnote{1also known as de Moivre’s Theorem.}
(d) In the complex plane, sketch the set of all the complex numbers such that $|z| = 1$. What are the $z$ values where the sketched figure intersects the real axis and the imaginary axis?

**Solution:**

![Diagram showing a circle with radius 1 in the complex plane.]  

(e) If $z = re^{j\theta}$, prove that $\bar{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

**Solution:**

\[
\bar{z} = (r\cos(\theta) + j\sin(\theta)) = r\cos(\theta) - j\sin(\theta) = r\cos(-\theta) + j\sin(-\theta) = re^{-j\theta}
\]

(f) Show (by direct calculation) that,  
\[r^2 = z\bar{z}.
\]

**Solution:**

\[
z\bar{z} = re^{j\theta}re^{-j\theta} = r^2e^{j0}e^{-j0} = r^2e^0 = r^2
\]

4. RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:
Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short. For this problem, we use the same notations as in Problem 1.

(a) Suppose $R = 0 \text{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1 \text{ volt}$. **Find the initial conditions for $\tilde{x}(0)$**. Recall that $\tilde{x}$ is in the changed “nice” eigenbasis coordinates from the first problem.

**Solution:** Under these conditions, we can solve for $\lambda = \pm j \sqrt{\frac{1}{LC}} = \pm j \sqrt{\frac{1}{25 \times 10^{-15}}} = \pm j 2 \times 10^6$

$\lambda_1 = j 2 \times 10^6$, $\lambda_2 = -j 2 \times 10^6$

Using the rule we derived earlier for finding $V$, we have

$$V = \begin{bmatrix} 1 & 1 \\ -50j & 50j \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 0.5 & 0.01j \\ 0.5 & -0.01j \end{bmatrix}$$

which lets us say

$$\tilde{x}(0) = V^{-1} \tilde{x}(0) = \begin{bmatrix} 0.5 & 0.01j \\ 0.5 & -0.01j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01j \\ -0.01j \end{bmatrix}$$

(b) Continuing the previous part, **find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$**.

**Solution:**

Plugging in for the component values gives:

$$\tilde{A} = \begin{bmatrix} j 2 \times 10^6 & 0 \\ 0 & -j 2 \times 10^6 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt} \tilde{x}_1(t) \\ \frac{d}{dt} \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} j 2 \times 10^6 & 0 \\ 0 & -j 2 \times 10^6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \quad (21)$$

Therefore:
\[\begin{align*}
\tilde{x}_1(t) &= K_1 e^{j2 \times 10^6 t} \\
\tilde{x}_2(t) &= K_2 e^{-j2 \times 10^6 t}
\end{align*}\]

Solving for \(K\) with the initial condition gives:

\[\begin{align*}
\tilde{x}_1(t) &= 0.01 j e^{j2 \times 10^6 t} \\
\tilde{x}_2(t) &= -0.01 j e^{-j2 \times 10^6 t}
\end{align*}\]

Converting back to the \(\tilde{x}\) coordinates:

\[\begin{align*}
\tilde{x}(t) &= V \tilde{x}(t) = \begin{bmatrix} 1 & 1 \\ -50j & 5 \times 50j \end{bmatrix} \begin{bmatrix} 0.01 j e^{j2 \times 10^6 t} \\ -0.01 j e^{-j2 \times 10^6 t} \end{bmatrix} \\
x_1(t) &= 0.01 j e^{j2 \times 10^6 t} - 0.01 j e^{-j2 \times 10^6 t} = -0.02 \sin(2 \times 10^6 t) \\
x_2(t) &= 0.5 e^{j2 \times 10^6 t} + 0.5 e^{-j2 \times 10^6 t} = \cos(2 \times 10^6 t)
\end{align*}\]

(c) Continuing the previous part, are the waveforms for \(x_1(t)\) and \(x_2(t)\) “transient” — do they die out with time?

Note: Because there is no resistance, this is called the “undamped” case.

Solution: No, these waveforms are sinusoids and do not die out over time. They are not transient.

5. RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:

Assume the circuit above has reached steady state for \(t < 0\). At time \(t = 0\), the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

(a) Now suppose that \(R = 1\ \Omega\) and the other component values are as specified in the circuit. Assume that \(V_s = 1\ \text{Volt}\). \textbf{Find the initial conditions for } \tilde{x}(0). \text{ Recall that } \tilde{x} \text{ is in the changed “nice” eigenbasis coordinates from the first problem.}
Solution: Under these conditions, we can solve for
\[ \lambda_1 = -0.02 \times 10^6 + j2 \times 10^6, \quad \lambda_2 = -0.02 \times 10^6 - j2 \times 10^6 \]
\[ V = \begin{bmatrix} 1 & 0 \\ -0.0002 & -0.0002 \end{bmatrix} \]
\[ \frac{x(t)}{\bar{x}(t)} = V^{-1}\bar{x}(t) = \begin{bmatrix} j0.010001 \\ -j0.010001 \end{bmatrix} \]

(b) Continuing the previous part, find \( x_1(t) = I_L(t) \) and \( x_2(t) = V_C(t) \) for \( t \geq 0 \).
(HINT: Remember that \( e^{a+jb} = e^a e^{jb} \).

Solution:
\[ \bar{A} = \begin{bmatrix} -0.02 \times 10^6 + j2 \times 10^6 & 0 \\ 0 & -0.02 \times 10^6 - j2 \times 10^6 \end{bmatrix} \]
\[ \begin{bmatrix} \frac{d}{dt}\bar{x}_1(t) \\ \frac{d}{dt}\bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.02 \times 10^6 + j2 \times 10^6 & 0 \\ 0 & -0.02 \times 10^6 - j2 \times 10^6 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \quad (22) \]
Therefore:
\[ \bar{x}_1(t) = K_1e^{(-0.02\times10^6+j2\times10^6)t} \]
\[ \bar{x}_2(t) = K_2e^{(-0.02\times10^6-j2\times10^6)t} \]

Solving for \( K \) with the initial condition gives:
\[ \bar{x}_1(t) = j0.010001e^{(-0.02\times10^6+j2\times10^6)t} \]
\[ \bar{x}_2(t) = -j0.010001e^{(-0.02\times10^6-j2\times10^6)t} \]

Converting back to the \( \bar{x} \) coordinates:
\[ \bar{x}(t) = V\bar{x}(t) = \begin{bmatrix} 1 & 1 \\ -0.49995 - j49.995 & -0.49995 + j49.995 \end{bmatrix} \bar{x}(t) \]
\[ x_1(t) = j0.010001e^{(-0.02\times10^6+j2\times10^6)t} - j0.010001e^{(-0.02\times10^6-j2\times10^6)t} \]
\[ x_2(t) = (0.5 - j0.005)e^{(-0.02\times10^6+j2\times10^6)t} + (0.5 + j0.005)e^{(-0.02\times10^6-j2\times10^6)t} \]
\[ x_1(t) = j0.010001e^{-0.02\times10^6}e^{j2\times10^6t} - j0.010001e^{-0.02\times10^6}e^{-j2\times10^6t} \]
\[ x_2(t) = (0.5 - j0.005)e^{-0.02\times10^6}e^{j2\times10^6t} + (0.5 + j0.005)e^{-0.02\times10^6}e^{-j2\times10^6t} \]
\[ x_1(t) = e^{-0.02\times10^6} \left( j0.010001e^{j2\times10^6t} - j0.010001e^{-j2\times10^6t} \right) = -0.020002e^{-0.02\times10^6} \sin(2 \times 10^6t) \]
\[ x_2(t) = e^{-0.02\times10^6} \left( (0.5 - j0.005)e^{j2\times10^6t} + (0.5 + j0.005)e^{-j2\times10^6t} \right) \]
\[ = e^{-0.02\times10^6} \cos(2 \times 10^6t) + 0.01 \cdot e^{-0.02\times10^6} \sin(2 \times 10^6t) \].
(c) Continuing the previous part, are the waveforms for \( x_1(t) \) and \( x_2(t) \) “transient” — do they die out with time?

Note: Because the resistance is so small, this is called the “underdamped” case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don’t have enough damping.

**Solution:** Yes, the waveforms are transient. There is a decaying exponential multiplied by the sinusoidal signal which causes the waveforms to die out over time.

(d) Notice that you got answers in terms of complex exponentials. **Why did the final voltage and current waveforms end up being purely real?**

**Solution:** In this case, it’s because of the complex conjugacy of the quantities in the problem. The eigenvalues and their associated eigenvectors were complex conjugates, as were the transformed solutions \( \tilde{x}_1(t) \) and \( \tilde{x}_2(t) \). When we applied the inverse transformation to \( \tilde{x}_1(t) \) and \( \tilde{x}_2(t) \), we added together many complex conjugate terms, and the imaginary parts cancelled out.

Now, was this just a fluke that just happened to line up perfectly? Is there some \( A \) matrix out there with real-valued entries that will result in a complex solution? Or is something more profound going on?

It turns to be no fluke. If the entries in the \( A \) matrix are real, and the initial condition \( \tilde{x}_0 \) is real, then the solution to the differential equation \( \frac{d}{dt} \tilde{x} = A \tilde{x} \) with \( \tilde{x}(0) = \tilde{x}_0 \) will also be real, regardless of whether the eigenvalues of \( A \) are real, imaginary, or complex. If a matrix \( A \in \mathbb{R}^{n \times n} \) has some complex eigenvalues, then those eigenvalues will always arise in complex conjugate pairs. Furthermore, the eigenvectors associated to those eigenvalues arise on complex conjugate pairs. This will lead to the kind of cancellation that you saw in here, every single time.

In a way, we could have predicted this. After all, the quantities that we observe in the world are always purely real, so we would expect that the solutions to our models would also be real-valued.

6. RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice \( R \) is not specified yet. You’ll have to figure out what that is.)

Assume the circuit above has reached steady state for \( t < 0 \). At time \( t = 0 \), the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

(a) **For what value of \( R \) is there going to be a single eigenvalue of \( A \)?**
Solution: If the terms under the square root, i.e., the discriminant of the quadratic formula, is 0, then we have a single value. More concretely,

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0$$  \hspace{1cm} (23)$$

$$\Rightarrow \frac{R^2}{L^2} = \frac{4}{LC}$$  \hspace{1cm} (24)$$

$$\therefore R = 2\sqrt{\frac{L}{C}}$$  \hspace{1cm} (25)$$

(b) When there is a single eigenvalue of this particular matrix $A$, what is the dimensionality of the corresponding eigenspace? (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?) For this part, assume the given values for the capacitor and the inductor, as well as the critical value for the resistance $R$ that you found in the previous part. It is easier to do the algebra with a non-symbolic matrix to work with.

Solution: Our system’s matrix becomes,

$$A = \begin{bmatrix} -4 \times 10^6 & -4 \times 10^4 \\ 10^8 & 0 \end{bmatrix}$$  \hspace{1cm} (26)$$

Our single eigenvector is,

$$\lambda = -\frac{R}{2L} = -2 \times 10^6$$  \hspace{1cm} (27)$$

Hence, the eigenvector is a basis of the nullspace of $A - \lambda I$,

$$\begin{bmatrix} -2 \times 10^6 & -4 \times 10^4 \\ 10^8 & 2 \times 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (28)$$

Hence, the eigenvector, $\vec{v} = \alpha \begin{bmatrix} 1 \\ -50 \end{bmatrix}$. We have only one eigenvector, since the we have a single dimensional nullspace.

(c) For a new coordinate system $V$, pick the first vector as being $\vec{v}_\lambda$ — the eigenvector you found for the single eigenvalue $\lambda$ above. For the second vector, just pick $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This implicitly defines variables $\vec{\tilde{x}}$ in the transformed coordinates so that $\vec{x}(t) = V\vec{\tilde{x}}(t)$. What is the resulting $\tilde{A}$ matrix defining the system of differential equations in the transformed coordinates?

Solution: We want to find $\tilde{A}$ such that,

$$\frac{d}{dt} \vec{\tilde{x}} = A\vec{\tilde{x}}$$  \hspace{1cm} (29)$$

$$\Rightarrow \frac{d}{dt} V\vec{\tilde{x}} = AV\vec{\tilde{x}}$$  \hspace{1cm} (30)$$

$$\Rightarrow V \frac{d}{dt} \vec{\tilde{x}} = AV\vec{\tilde{x}}$$  \hspace{1cm} (31)$$

$$\therefore \frac{d}{dt} \vec{\tilde{x}} = V^{-1}AV\vec{\tilde{x}}$$  \hspace{1cm} (32)$$
Hence, we have

\[
\tilde{A} = V^{-1}AV = \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \times 10^6 & -4 \times 10^4 \\ 1 \times 10^8 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -50 & 1 \end{bmatrix}
\]

(33)

\[
= \begin{bmatrix} -2 \times 10^6 & -4 \times 10^4 \\ 0 & -2 \times 10^6 \end{bmatrix}
\]

(34)

(d) Notice that the second differential equation for \( \frac{d}{dt}\tilde{x}_2(t) \) in the above coordinate system only depends on \( \tilde{x}_2(t) \) itself. There is no cross-term dependence. **Compute the initial condition for \( \tilde{x}_2(0) \) and write out the solution to this scalar differential equation for \( \tilde{x}_2(t) \) for \( t \geq 0 \).**

**Solution:** First let’s compute \( \tilde{x}_2(0) \),

\[
\begin{bmatrix} \tilde{x}_1 \end{bmatrix} = V^{-1} \begin{bmatrix} x_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\( \therefore \tilde{x}_2(0) = 1 \) \( \ldots \) (37)

Solving the following differential equation,

\[
\frac{d}{dt}\tilde{x}_2 = -2 \times 10^6 \tilde{x}_2
\]

\( \therefore \tilde{x}_2 = k_1 e^{-2 \times 10^6 t} \)

Substituting for the initial condition, we get, \( \tilde{x}_2 = e^{-2 \times 10^6 t} \)

(e) With an explicit solution to \( \tilde{x}_2(t) \) in hand, substitute this in and **write out the resulting scalar differential equation for \( \tilde{x}_1(t) \).** This should effectively have an input in it.

Note: this is just the differential-equations counterpart to the back-substitution step that you remember from learning Gaussian Elimination in 16A, once you had done one full downward pass of Gaussian Elimination. You went upwards and just substituted in the solution that you found to remove this dependence from the equations above. This is the exact same design pattern, except for a system of linear differential equations.

**Solution:** The differential equation for \( \frac{d}{dt}\tilde{x}_1(t) \) is

\[
\frac{d}{dt}\tilde{x}_1(t) = \left(-2 \times 10^6\right)\tilde{x}_1(t) - \left(4 \times 10^4\right)\tilde{x}_2(t),
\]

(38)

which is just the top row of the matrix equation \( \frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) \). Substituting in the solution we found for \( \tilde{x}_2(t) \) gives

\[
\frac{d}{dt}\tilde{x}_1(t) = \left(-2 \times 10^6\right)\tilde{x}_1(t) - \left(4 \times 10^4\right)e^{(-2 \times 10^6)t}.
\]

(39)

Just like we expected, this is a scalar differential equation with an input.

(f) **Solve the above scalar differential equation with input and write out what \( \tilde{x}_1(t) \) is for \( t \geq 0 \).**

*(HINT: You might want to look at a problem on an earlier homework for help with this.)*

**Solution:**
Recall from an earlier part of this homework, that the differential equation
\[ \frac{dx(t)}{dt} = \lambda x(t) + \beta e^{\lambda t}, \] (40)
with initial value \( x(t) = x_0 \), has the solution
\[ x(t) = x_0 e^{\lambda t} + \beta te^{\lambda t}. \] (41)

If you haven’t already verified the solution of this differential equation as part of this assignment, please do so now.

With that said, the differential equation we found for \( \tilde{x}_1(t) \) in the previous part is of this form, with \( \lambda = -2 \times 10^6, \beta = -4 \times 10^4, \) and \( x_0 = 0, \) so we know the solution is
\[ \tilde{x}_1(t) = -\left(4 \times 10^4\right)te^{(-2\times10^6)t}. \] (42)

(g) Find \( x_1(t) \) and \( x_2(t) \) for \( t \geq 0 \) based on the answers to the previous three parts.

This particular case is called the “critically damped case” for an RLC circuit. It is called this because the \( R \) value you found demarcates the boundary between solutions of the underdamped and overdamped variety.

**Solution:** Now that we have \( \tilde{x}_1(t) \) and \( \tilde{x}_2(t) \), all we need to do to find \( x_1(t) \) and \( x_2(t) \) is to reverse the coordinate change we made. In other words, we can find \( x(t) \) as
\[ x(t) = V\tilde{x}(t). \] (43)

This gives
\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
-50 & 1
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1(t) \\
\tilde{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
-50 & 1
\end{bmatrix}
\begin{bmatrix}
-\left(4 \times 10^4\right)te^{(-2\times10^6)t}
\\
e^{(-2\times10^6)t}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-\left(4 \times 10^4\right)te^{(-2\times10^6)t} \\
\left(1 + 2 \times 10^6t\right)e^{(-2\times10^6)t}
\end{bmatrix}
\] (44)

(h) To see the impact of changing the parameters \( R \) and \( C \), play with the included Jupyter notebook. See what happens above and below the critically damped condition. **Comment on what you observed.**

Note: The curve evaluation code in the included notebook has been slightly obfuscated from the approach taken in the parts above for solving the differential equations. So, it is not really going to be that useful to read those details of the code. You are, of course, free to try out your own expressions by editing the code as well as to add plots for individual parts — like separating out the contributions that are coming from each of the underlying modes for this circuit (i.e. the contribution coming from each of the eigenvalues).

**Solution:** We observe that as we approach the critical value of \( R = 100 \), the oscillations get damped down faster and faster as we increase \( R \). But then, when we keep going to higher \( R \), the oscillations cease and the plot goes more slowly to 0. The critically damped value for \( R \) seems to be in the neighborhood of the fastest drop to near zero.

This can be justified mathematically by showing that the maximum real part of the eigenvalues is minimized at the point where the circuit is critically damped — i.e. at this point, the real part of the eigenvalues is made as negative as possible at this point.
7. **Alternative “second order” perspective on solving the RLC circuit**

Consider the following circuit like you saw in lecture, discussion, and the previous few problems:

![RLC Circuit Diagram]

Suppose now we insisted on expressing everything in terms of one waveform \( V_C(t) \) instead of two of them (voltage across the capacitor and current through the inductor). This is called the “second-order” point of view, for reasons that will soon become clear.

For this problem, use \( R \) for the resistor, \( L \) for the inductor, and \( C \) for the capacitor in all the expressions until the last part.

(a) **Write the current \( I_L \) through the inductor in terms of the voltage through the capacitor.**

**Solution:** The current \( I_L \) through the inductor must be the same as the current through \( C \), which is \( C \frac{d}{dt} V_C \). Hence, we can write

\[
I_L = C \frac{d}{dt} V_C.
\]

(b) Now, notice that the voltage drop across the inductor involves \( \frac{d}{dt} I_L \). **Write the voltage drop across the inductor in terms of the second derivative of \( V_C \).**

**Solution:** The voltage drop is

\[
V_L = L \cdot \frac{d}{dt} I_L = LC \frac{d}{dt} \left( \frac{d}{dt} V_C \right) = LC \frac{d^2}{dt^2} V_C.
\]

(c) For this part, treat \( V_s(t) \) as a generic input waveform — don’t necessarily view the switch as being thrown, etc.

**Now write out a differential equation governing \( V_C(t) \) in the form of**

\[
\frac{d^2}{dt^2} V_C(t) + a \cdot \frac{d}{dt} V_C(t) + b \cdot V_C(t) + c(t) = 0.
\]

(46)

where \( a, b \) and \( c(t) \) are terms you need to figure out by analyzing the circuit.

*(HINT: The \( c(t) \) needs to involve \( V_s(t) \) in some way.)*

**Solution:** Note that the current passing through the resistor is

\[
I_R = \frac{V_s - V_C - V_L}{R} = C \frac{d}{dt} V_C.
\]

or equivalently,

\[
V_L + RC \frac{d}{dt} V_C + V_C - V_s = 0.
\]
Plugging in $V_L$, we have
\[ LC \frac{d^2}{dt^2} V_C + RC \frac{d}{dt} V_C + V_C - V_s = 0. \]

Finally, dividing by $LC$,
\[ \frac{d^2}{dt^2} V_C + \frac{R}{L} \cdot \frac{d}{dt} V_C + \frac{1}{LC} \cdot V_C - \frac{1}{LC} \cdot V_s = 0. \]

(d) We don’t know how to solve equations like Eq. (46). To reduce this to something we know how to solve, we define $X(t)$ as an additional state, with $X(t) = \frac{d}{dt} V_C(t)$. Note that this directly gives us one equation: $\frac{d}{dt} V_C(t) = X(t)$. This leaves us needing an equation for $\frac{d}{dt} X(t)$. **Express $\frac{d}{dt} X(t)$ in terms of $X(t)$, $V_C(t)$, and $V_s(t)$**. Write a matrix differential equation in terms of $V_C(t)$ and $X(t)$. Your answer should be in the form:
\[
\frac{d}{dt} \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} X(t) \\ V_C(t) \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} \cdot V_s(t).
\]

(47)

**Solution:** With our expression of $X$, we can write
\[
\frac{d}{dt} X + \frac{R}{L} \cdot X + \frac{1}{LC} \cdot V_C - \frac{1}{LC} \cdot V_s = 0
\]
and
\[
\frac{d}{dt} V_C = X.
\]

Then, we can write a matrix
\[
\begin{bmatrix}
\frac{d}{dt} X \\
\frac{d}{dt} V_C
\end{bmatrix} = \begin{bmatrix}
-\frac{R}{L} & -\frac{1}{LC} \\
1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
X \\
V_C
\end{bmatrix} + \begin{bmatrix}
\frac{1}{LC} \\
0
\end{bmatrix} \cdot V_s.
\]

(e) **Find the eigenvalues and eigenvectors of the matrix $A$ from Eq. (47).**

(Hint: use the same trick you did in problem 1. Don’t do this the hard way.)

**Solution:** $\det(A - \lambda I) = 0$ gives us
\[
\lambda^2 + \frac{R}{L} \cdot \lambda + \frac{1}{LC} = 0.
\]

Then,
\[
\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}.
\]

To find the eigenvectors corresponding to $\lambda$, we assume the eigenvector is of the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$. Then,
\[
1 = \lambda \cdot a \implies a = \frac{1}{\lambda}, \text{ and hence, we conclude that}
\]
\[
\begin{align*}
\text{Eigenvalue } \frac{-R}{L} + \frac{\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} & \text{ corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{1}{\lambda} \end{bmatrix} \\
\text{Eigenvalue } \frac{-R}{L} - \frac{\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2} & \text{ corresponds to eigenvector } \begin{bmatrix} 1 \\ \frac{1}{\lambda} \end{bmatrix}.
\end{align*}
\]
(f) Revisit Problems 2 and 5, and use the values of $R, L, C$, and the same initial conditions to solve for $V_C$. Did you get the same answer as in problems 2 and 5?

Solution: Let’s first solve the differential equation

$$\begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{RC} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ V_C \end{bmatrix}.$$ 

For problem 2, the eigenvalues are $\lambda_1 = -1.0 \times 10^5$ and $\lambda_2 = -4.0 \times 10^7$, with

$$V^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}^{-1} = \frac{1}{\lambda_1} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ -\frac{1}{\lambda_1} & 1 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{-\lambda_1\lambda_2}{\lambda_1-\lambda_2} \\ \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{-\lambda_1\lambda_2}{\lambda_1-\lambda_2} \end{bmatrix}.$$ 

For ease of notations, let $\vec{y} := \begin{bmatrix} X \\ V_C \end{bmatrix}$. The initial conditions are

$$\vec{y}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix} -0.0025 & 1.0 \times 10^5 \\ 1.0 \times 10^5 & -1.0 \times 10^5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.0 \times 10^5 \\ -1.0 \times 10^5 \end{bmatrix}.$$ 

This implies that

$$\vec{y} = \begin{bmatrix} 1.0 \times 10^5 \\ 0 \\ -1.0 \times 10^5 \end{bmatrix} \begin{bmatrix} e^{-1.0 \times 10^5 t} \\ e^{-4.0 \times 10^7 t} \end{bmatrix} = \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} \\ -1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}.$$ 

Finally,

$$\vec{y}(t) = V\vec{y} = \begin{bmatrix} 1 \\ -1.0 \times 10^{-5} \\ -2.5 \times 10^{-8} \end{bmatrix} \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} \\ -1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}$$

and hence

$$\vec{y}(t) = \begin{bmatrix} 1.0 \times 10^5 \cdot e^{-1.0 \times 10^5 t} - 1.0 \times 10^5 \cdot e^{-4.0 \times 10^7 t} \\ -1.0 \times e^{-1.0 \times 10^5 t} + 2.5 \times 10^{-3} \cdot e^{-4.0 \times 10^7 t} \end{bmatrix}.$$ 

Comparing $V_C(t)$, we see that we get the same answer as in problem 2.

For problem 5, the eigenvalues are $\lambda_1 = -2.0 \times 10^4 + j2 \times 10^6$ and $\lambda_2 = -2.0 \times 10^4 - j2 \times 10^6$, with

$$V^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \end{bmatrix}^{-1} = \frac{1}{\lambda_1} \begin{bmatrix} \frac{1}{\lambda_2} & -1 \\ -\frac{1}{\lambda_1} & 1 \end{bmatrix} = \begin{bmatrix} \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{-\lambda_1\lambda_2}{\lambda_1-\lambda_2} \\ \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{-\lambda_1\lambda_2}{\lambda_1-\lambda_2} \end{bmatrix}.$$ 

For ease of notations, let $\vec{y} := \begin{bmatrix} X \\ V_C \end{bmatrix}$. The initial conditions are

$$\vec{y}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix} -0.0005 + j0.5 & j10^6 \\ 0.0005 + j0.5 & -j10^6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j10^6 \\ -j \times 10^6 \end{bmatrix}.$$ 

This implies that

$$\vec{y} = \begin{bmatrix} j10^6 \cdot e^{(-2.0 \times 10^4 \pm j2 \times 10^6) t} \\ -j10^6 \cdot e^{(-2.0 \times 10^4 \mp j2 \times 10^6) t} \end{bmatrix}.$$
Finally,
\[ \vec{y}(t) = V \vec{\tilde{y}} = \begin{bmatrix} 1 & 1 \\ -2.0 \times 10^4 + j2 \times 10^6 & -2.0 \times 10^4 - j2 \times 10^6 \end{bmatrix} \begin{bmatrix} j10^6 \cdot e^{(-2.0 \times 10^4 + j2 \times 10^6)t} \\ j10^6 \cdot e^{(-2.0 \times 10^4 - j2 \times 10^6)t} \end{bmatrix} \]
and hence
\[ \vec{y}(t) = \begin{bmatrix} j10^6 \cdot e^{(-2.0 \times 10^4 + j2 \times 10^6)t} \\ j10^6 \cdot e^{(-2.0 \times 10^4 - j2 \times 10^6)t} \end{bmatrix} \begin{bmatrix} -2.0 \times 10^4 + j2 \times 10^6 \\ -2.0 \times 10^4 - j2 \times 10^6 \end{bmatrix} \]

Writing out \( V_C(t) \), we can simplify it approximately to
\[ V_C(t) = e^{-0.02 \times 10^6 t} \cos(2 \times 10^6 t) + 0.01 \cdot e^{-0.02 \times 10^6 t} \sin(2 \times 10^6 t). \]

Comparing \( V_C(t) \), we see that we get the same answer as in problem 5.

8. A toy model for a solar cell (PRACTICE)

In 16A’s imaging labs, you used an electronic component that responded to light in a way that could be detected electrically. To truly properly understand such things, you need to take courses like EE130 and EE134. However, in this problem, we will walk you through the modeling of a heavily simplified caricature of such a device.

In Figure 2, we illustrate what is effectively one-half of a solar cell. In simple English, what is happening is that light is striking the device and constantly causing free electron/hole pairs to be created (think of this as a kind of puddle of free charge carriers). On one side (depicted here), the electrons end up diffusing through the material until they reach a metal wire, at which point they run through the (not shown) attached circuit to meet their counterpart holes on the other side of the solar cell. The other half is symmetric, except dealing with holes. EECS 16B is a course without Physics prerequisites, and so the detailed nature of the physics here is out of scope. However, we would like to see the connection between the density of charge carriers being created by the light and the current that flows out of the solar cell.

Again, this problem is a vastly simplified caricature of what is going on in the real world, but it allows us to both get a feeling for what is happening as well as practice many core skills in 16B.

The most fundamental thing in this problem is to look at the steady state distribution of free charge carriers in the following setup, depicted in Figure 2.

Figure 2: A hypothetical half of a toy model for a solar cell. At \( x = 0 \), where the light is liberating charge, the density \( q \) of free charge carriers is held constant at \( C \) which depends on the intensity of the incident light. Meanwhile, at \( x = L \), the density of free charge carriers is held constant at 0 because they get whisked away by the conducting metal wire — to race around the circuit to be reunited with their separated partners.
Let us define \( q(x,t) \) as the density of charge at point \( x \) at time \( t \).

Although we are interested in the steady-state behavior where things are going to end up not depending on \( t \), the potential dependence on \( t \) is important to understand the differential equations that govern the behavior of the system.

In the above figure (Fig. 2), a plate with special material at \( x = 0 \) generates free charge carriers from light\(^2\). The plate is exposed to a constant light source so that the charge density \( q(0,t) \) is held constant at \( C \). On the rightmost end (\( x = L \)), a metal plate connected to a circuit that loops back to the other side of the solar cell forces the charge density at \( x = L \) to be \( q(L,t) = 0 \) at all times.

Our goal is to understand what happens for the rest of \( q(x,t) \) for \( 0 < x < L \).

To do so, we need to understand the dynamics that govern the behavior of charge density in the middle of this material. Physically, what is going on? What’s happening is that the free charge carriers are just wandering around randomly in the material. In our simplified toy model here, they have no reason to prefer moving right or left and any individual free charge carrier is just as likely to move in one direction as the other. Such random motion is called diffusion. How can we translate this into a differential equation with some predictive power?

To understand this, let’s see what happens in the hypothetical small box between lengths \( x = s \) and \( x = s + ds \) at time \( t \). It turns out that an important quantity that we would like to understand is the gradient \( g(x,t) = \frac{d}{dx} q(x,t) \).

Due to the nature of random motion, in a small time \( dt \), the amount of charge flowing into the box from the left \( x = s \) side is equal to \(-K \cdot g(s,t) \cdot dt\). Here, the constant \( K \) depends on various physical constraints. You can think of it as how freely the free charge carriers are allowed to run around in the material. (Why the minus sign? Because the random flow of charge opposes the gradient of charge density — it wants to make things more level. Think of shaking a pile of sand, it will want to become less uneven by randomly flowing down the pile, not up the pile.) Meanwhile, the amount of charge entering the box from the right side \( x = s + ds \) is equal to \( K \cdot g(s + ds,t) \cdot dt \). Hence, the change in the amount of charge that the small box gets is equal to

\[
(-K \cdot g(s,t) \cdot dt) + (K \cdot g(s + ds,t) \cdot dt) \approx K \cdot \left( \frac{d}{dx} g(x,t) \big|_{x = s} \right) \cdot ds \cdot dt.
\]

On the other hand, in a small amount of time \( dt \), the change in the amount of charge in the box is also

\[
\left( \frac{d}{dt} q(x,t) \big|_{x = s} \right) \cdot dt \cdot ds.
\]

These must be the same, and so we can equate the two expressions to get:

\[
K \cdot \left( \frac{d}{dx} g(x,t) \big|_{x = s} \right) = \frac{d}{dt} q(x,t) \big|_{x = s}.
\]

Since this holds for all \( s \) and all times \( t \), it follows that

\[
\frac{d}{dt} q(x,t) = K \frac{d}{dx} g(x,t)
\]

for some constant \( K \) that depends on the material and other physics constants.

\(^2\)An example is an appropriate PN junction for a solar cell. Take 130 and/or 134 for more information!
As we can see, our knowledge of differential equations allows us to write down such a model. Equation (49) is sometimes referred to as the heat equation since it also models heat flow. (The role of the charge density is played by the temperature.)

In this problem, we are only interested in the steady-state case, i.e., we are going to assume that \( q(x, t) \) does not change over time. That implies \( \frac{d}{dt} q(x, t) = 0 \) and using that, we can simplify our expression of \( q(x, t) \) and write it as \( q(x) \). Consequently, we can also simplify \( g(x, t) \) to \( g(x) \). Now, solving Equations (48) and (49) is equivalent to solving something we are familiar with:

\[
\frac{d}{dx} q(x) = g(x), \\
\frac{d}{dx} g(x) = 0.
\]  

This is a system of differential equations of the type we know how to handle. So let’s solve for both \( q(x) \) and \( g(x) \) from Equations (50) and (51).

(a) **Write out the differential equation in matrix/vector form:**

\[
\frac{d}{dx} \begin{bmatrix} q(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \cdot \begin{bmatrix} q(x) \\ g(x) \end{bmatrix}
\]

Here, the “??” in the expression above simply represent the entries of the \( A \) matrix. That’s what you need to fill in.

**Solution:** Note that \( \frac{d}{dx} q(x) = g(x) \) and \( \frac{d}{dx} g(x) = 0 \). Hence, \( A \) can be written as

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

(b) **Find the eigenvalues and eigenvectors of \( A \).**

**Solution:** Note that

\[
\text{det}(A - \lambda \cdot I) = \lambda^2.
\]

Solving \( \text{det}(A - \lambda \cdot I) = 0 \), we see that both of the eigenvalues of \( A \) are 0. The eigenvectors corresponding to eigenvalue 0 are then \( k \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

Notice that this problem only has a one-dimensional eigenspace despite being a 2x2 matrix. This is like what we saw in the critically damped case for the RLC circuit. Such situations are common in engineering settings.

(c) Assume that we know both \( q(0) \) and \( g(0) \). **Solve for \( q(x) \) and \( g(x) \) in terms of these initial conditions.** *(Hint: think about what you did for the critically damped case for the RLC circuit.)*

**Solution:**

The key idea in the critically damped RLC case was not to panic and just to proceed systematically. What do we know how to solve?

First, notice that \( \frac{d}{dx} g(x) = 0 \) implies that \( g(x) = T \) for some constant \( T \). However, since we know \( g(0) \), this implies that \( g(x) = g(0) \). Then, we have

\[
\frac{d}{dx} q(x) = g(0)
\]
which implies that $q(x) = g(0) \cdot x + R$ for some constant $R$. Since we know $q(0)$, this implies

$$q(0) = g(0) \cdot 0 + R = R.$$ 

Hence, we conclude that $R = q(0)$, and that

$$q(x) = g(0) \cdot x + q(0).$$

(d) The challenge is that the physical story does not tell us anything immediately about $g(0)$. Instead, we just know about the free carrier density at both endpoints. **Solve for $q(x)$ and $g(x)$ with boundary conditions $q(0) = C$ and $q(L) = 0$ instead.**

(Hint: If you knew $g(0)$, what would $q(L)$ be in terms of $q(0)$ and $g(0)$? But you know $q(L)$ so what does that imply?)

**Solution:** First, notice that $\frac{d}{dx} g(x) = 0$ implies that $g(x) = T$ for some constant $T$. However, since we don’t know $g(0)$, we leave the $T$ to be solved in the future. Then, we have

$$\frac{d}{dx} q(x) = T$$

which implies that $q(x) = T \cdot x + R$ for some constant $R$. Since we know $q(0) = C$ and $q(L) = 0$, this implies

$$C = q(0) = T \cdot 0 + R.$$ 

Hence, we conclude that $R = C$, and that

$$0 = q(L) = T \cdot L + C$$

which implies $T = -\frac{C}{L}$. Thus, we conclude that

$$q(x) = -\frac{C}{L} \cdot x + C.$$ 

This is just a constant slope line that starts at $C$ and then reaches zero at $L$.

(e) The gradient $g(L)$ is related to the current flowing from the wire into the metal plate. **What is $g(L)$?**

**Solution:** This can be computed explicitly from our solution $q(x)$.

$$g(L) = \left. \frac{d}{dx} q(x) \right|_{x=L} = -\frac{C}{L}.$$ 

(f) The above is an extremely simplified model of what happens in a solar cell. To be more realistic, you could also model the random recombination of free charge carriers within the medium itself. This recombination is proportional to the local density of free charge carriers themselves and thus modifies (49) to instead be $\frac{d}{dt} q(x,t) = K \frac{d}{dx} g(x,t) - K_2 q(x,t)$. Because we still want $\frac{d}{dt} q(x,t) = 0$, this changes (51) to be $\frac{d}{dx} g(x) = \frac{K_2}{K} q(x)$ where $K_2 > 0$ is another physical constant that depends on the material. **What is your solution for $q(x)$ in this case?**

(HINT: It is convenient here to avoid having to calculate the eigenvectors at all. You know that the solution will have two terms to it — one corresponding to each of the distinct eigenvalues. Just get the eigenvalues and then fit to the boundary conditions.)
**Solution:**  Note that in this case the matrix $A$ for this problem is

$$A = \begin{bmatrix} 0 & 1 \\ \frac{K_2}{K} & 0 \end{bmatrix}.$$  

Writing out $\det(A - \lambda I) = 0$ gives us

$$\lambda^2 - \frac{K_2}{K} = 0$$

which implies

$$\lambda_1 = \sqrt{\frac{K_2}{K}}, \quad \lambda_2 = -\sqrt{\frac{K_2}{K}}.$$  

Here, it appears that one of the eigenvalues $\lambda_1$ is negative and stable (corresponding to an exponential that will decay) while the other is positive and unstable (corresponding to a growing exponential).  

Now, we write out the general solution for $q(x)$:

$$q(x) = a_1 \cdot e^{\sqrt{\frac{K_2}{K}} x} + a_2 \cdot e^{-\sqrt{\frac{K_2}{K}} x}.$$  

By our boundary conditions $q(0) = C$ and $q(L) = 0$, we solve for $a_1$ and $a_2$ via the following equations:

$$a_1 + a_2 = C,$$

$$a_1 \cdot e^{\sqrt{\frac{K_2}{K}} L} + a_2 \cdot e^{-\sqrt{\frac{K_2}{K}} L} = 0.$$  

which gives us the solutions

$$a_1 = \frac{C}{1 + e^{2\sqrt{\frac{K_2}{K}} L}} \quad \text{and} \quad a_2 = \frac{C \cdot e^{2\sqrt{\frac{K_2}{K}} L}}{1 + e^{2\sqrt{\frac{K_2}{K}} L}}.$$  

Consequently, we have

$$q(x) = \frac{C}{1 + e^{2\sqrt{\frac{K_2}{K}} L}} \cdot e^{\sqrt{\frac{K_2}{K}} x} + \frac{C \cdot e^{2\sqrt{\frac{K_2}{K}} L}}{1 + e^{2\sqrt{\frac{K_2}{K}} L}} \cdot e^{-\sqrt{\frac{K_2}{K}} x}.$$  

Finally, since $g(x) = \frac{d}{dx} q(x)$, we get the expression

$$g(x) = \frac{C \cdot \sqrt{\frac{K_2}{K}} \cdot e^{\sqrt{\frac{K_2}{K}} x} - C \cdot e^{2\sqrt{\frac{K_2}{K}} L} \cdot \sqrt{\frac{K_2}{K}} \cdot e^{-\sqrt{\frac{K_2}{K}} x}}{1 + e^{2\sqrt{\frac{K_2}{K}} L}}.$$  

When the charge recombination rate $K_2$ is very small, then the above $q(x)$ ends up looking very much like the straight line solution that we got earlier and the current flowing out of the solar cell is large.  If the recombination rate is significant, the charge-carrier density dies more quickly with distance $x$ and the current flowing out of the solar cell is reduced.  

If you want to learn more about these kinds of things, take 130 and 134.  (After having taken some Physics!)  Device physics is a beautiful subject in which physical intuition plus quantum mechanics combine with differential equations and careful experimentation (for which, 143 is a great introduction — you will literally fabricate your own devices in the clean room) to give rise to the fundamental devices that are the foundation of our contemporary information era.
Whether we’re interested in providing cheap renewable energy to everyone on this planet, in building cybernetic implants to help people with disabilities, or in creating faster and more energy efficient computing platforms for machine intelligences — advancements in device physics lead the way. For almost \( \frac{3}{4} \) of a century now, our ability to solve problems has improved exponentially with most of that improvement coming from improved devices, another large fraction coming from improved theory/algorithms, and the rest dominated by our ability to scale to large-scale systems like cloud computing and the like. Along the way, computer architectures, analog circuits, and programming languages/platforms need to constantly adapt to leverage these underlying advances and enable them to keep working together.

9. Op-Amp Integrators: A continuation from the previous HW (PRACTICE)

In this question we will continue on from our analysis in Homework 2 and look at the eigenvalues of the integrator circuit (refer to Figure 5) in both non-ideal and ideal situations.

![Op-amp model: \( \Delta V = V_+ - V_- \)](image)

![Buffer in negative feedback](image)

![“Buffer” in positive feedback that doesn’t actually work as a buffer.](image)

Figure 3: Op-amp model: \( \Delta V = V_+ - V_- \)

Figure 4: Op-amp in buffer configuration
(a) Recall from Homework 2 we had the following analysis to the integrator circuit shown in Figure 6.

\[
\frac{d}{dt} \begin{bmatrix} V_{out} \\ V_C \end{bmatrix} = \begin{bmatrix} -\left( \frac{G+1}{\frac{1}{R_C}} + \frac{1}{R_{out}} \right) & -\left( \frac{1}{R_{out}} + \frac{G}{\frac{1}{R_C}} \right) \\ \frac{1}{R_C} & 0 \end{bmatrix} \begin{bmatrix} V_{out} \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{R_C} \\ \frac{C_{out}}{R_C} \end{bmatrix} V_{in} \tag{52}
\]

Solve for the eigenvalues for the matrix/vector differential equation in Eq. (52).

For simplicity, assume \( C_{out} = C = 0.01F \) and \( R = 1\Omega \) and looking at the datasheet for the TI LMC6482 (the op-amps used in lab), we have \( G = 10^6 \) and \( R_{out} = 100\Omega \).

Feel free to assume \( G+1 \approx 10^6 \) when you finally need to plug in values, but do not make any other approximations. (Of course, such an approximation is not valid if you have a \( G+1 - G \) term showing up somewhere.) Feel free to use a scientific calculator or Jupyter to find the eigenvalues.

**Solution:** We can find the characteristic equation and substituting the given values, we get

\[
\lambda^2 + \left( \frac{G+1}{\frac{1}{R_C}} + \frac{1}{R_{out}} \right) \lambda + \frac{1}{R_{out}C_{out}R_C} = 0
\]

\[
\lambda^2 + \left( 10^6 + 2 \times 10^2 \right) \lambda + 10^2 = 0
\]
Hence, the eigenvalues are

\[ \lambda_\pm = \frac{-(10^6 + 2 \times 10^2) \pm \sqrt{(10^6 + 2 \times 10^2)^2 - 400}}{2} \]

\[ \therefore \lambda_+ = -1 \times 10^{-4} \]
\[ \lambda_- = -1 \times 10^6 \]

You should see that one eigenvalue corresponds to a slowly dying exponential and is close to 0. The other corresponds to a much faster dying exponential. The very slowly dying exponential is what corresponds to the desired integrator-like behavior. This is what lets it “remember.” (If you don’t understand why, think back to the HW problem you saw in a previous HW where you proved the uniqueness of the integral-based solution to a scalar differential equation with an input waveform.)

(b) Again, assume we have an ideal op-amp, i.e., \( G \to \infty \). **Find the eigenvalues under this limit.** Feel free to make any reasonable approximations.

**Solution:** With the given assumptions, we can rewrite our characteristic equation as

\[ \lambda^2 + \frac{G}{R_{out}C_{out}} \lambda + \frac{1}{R_{out}C_{out}RC} = 0 \]

Hence, we can find the eigenvalues as

\[ \lambda_\pm = \frac{-G}{2R_{out}C_{out}} \pm \sqrt{\frac{G^2}{4R_{out}^2C_{out}^2} - \frac{1}{R_{out}C_{out}RC}} \]

\[ \therefore \lambda_+ \to 0 \]
\[ \lambda_- \to -\infty \]

Here, you should see that the eigenvalue that used to be a slowly dying exponential stops dying out at all — corresponding to the ideal integrator’s behavior of remembering forever.

10. **Write Your Own Question And Provide a Thorough Solution.**

Writing your own problems is a very important way to really learn material. The famous “Bloom’s Taxonomy” that lists the levels of learning is: Remember, Understand, Apply, Analyze, Evaluate, and Create. Using what you know to create is the top level. We rarely ask you any homework questions about the lowest level of straight-up remembering, expecting you to be able to do that yourself (e.g. making flashcards). But we don’t want the same to be true about the highest level. As a practical matter, having some practice at trying to create problems helps you study for exams much better than simply counting on solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really look at the material from the perspective of those who are going to create the exams. Besides, this is fun. If you want to make a boring problem, go ahead. That is your prerogative. But it is more fun to really engage with the material, discover something interesting, and then come up with a problem that walks others down a journey that lets them share your discovery. You don’t have to achieve this every week. But unless you try every week, it probably won’t ever happen.

11. **Homework Process and Study Group**

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.
(a) **What sources (if any) did you use as you worked through the homework?**

(b) **Who did you work on this homework with?** List names and student ID’s. (In case of homework party, you can also just describe the group.)

(c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

(d) **Roughly how many total hours did you work on this homework?**

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