1. Lagrange interpolation and polynomial basis

In practice, to approximate some unknown or complex function $f(x)$, we take n evaluations/samples of the function, denoted by $\{(x_i, y_i \triangleq f(x_i));\ 0 \leq i \leq n-1\}$. With the Occam's razor principle in mind, we try to fit a polynomial function of least degree (which is $n-1$) that passes through all the given points.

(a) Using the polynomial basis $\{1, x, x^2, \cdots, x^{n-1}\}$ studied in problem 1, the fitting problem can be cast into finding the coefficients $a_0, a_1, \cdots, a_{n-1}$ of the function

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

such that $g(x_i) = y_i,\ \forall i = 0, 1, \cdots n-1$. Find out the set of equations that need to be satisfied, and write them in a matrix form $A\vec{a} = \vec{y}$, with $\vec{a} = [a_0, a_1, \cdots, a_{n-1}]^T$ and $\vec{y} = [y_0, y_1, \cdots, y_{n-1}]^T$

(b) Now we observe that in order to find those coefficients, we need to calculate $\vec{a} = A^{-1}\vec{y}$. The matrix inversion is computationally expensive and numerically inaccurate when n is large. The idea of Lagrange interpolation is to use a different set of basis $\{L_0(x), L_1(x), \cdots, L_{n-1}(x)\}$, which has the property that

$$L_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

With that the fitting problem becomes finding the coefficients $b_0, b_1, \cdots, b_{n-1}$ of the function

$$h(x) = b_0 L_0(x) + b_1 L_1(x) + b_2 L_2(x) + \cdots + b_{n-1} L_{n-1}(x)$$

such that $h(x_i) = y_i,\ \forall i = 0, 1, \cdots n-1$. Again, find out the set of equations that need to be satisfied, and write them in a matrix form. What do you observer?

(c) Show that if we define

$$L_i(x) = \Pi_{j=0; j\neq i}^{j=n-1} \frac{(x - x_j)}{(x_i - x_j)}$$

then the property required in part (b) is satisfied. What is the intuition behind this construction?

(d) Based on the previous two parts, write down the explicit form of $h(x)$ with the samples $\{(x_i, y_i);\ 0 \leq i \leq n-1\}$. The resulting formula is the so called Lagrange polynomial which passes through the n sampled points.

(e) Find the Lagrange polynomial given evaluated samples $f(-1) = 3, f(0) = -4, f(1) = 5, f(2) = -6$.

2. Inner products of of polynomials

A polynomial of degree at most n on a single variable can be written as

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots p_n x^n$$

where we assume that the coefficients $p_0, p_1, \ldots, p_n$ are real. Let P_n be the vector space of all polynomials of degree at most n.

(a) We can define an inner product on P_2 by setting

$$\langle p,q\rangle = \sum_{i=0}^{2} p(i)q(i) = \begin{pmatrix} p(0) & p(1) & p(2) \end{pmatrix} \begin{pmatrix} q(0) \\ q(1) \\ q(2) \end{pmatrix}$$

This is equivalent to sampling the polynomials p,q at the points $0,1,2$ and taking the dot product of the resulting vectors. Show that this satisfies the following properties of a real inner product.

- $\langle p,p\rangle \geq 0$, with equality if and only if $p = 0$.
- For all $a \in \mathbb{R}$, $\langle ap,q\rangle = a\langle p,q\rangle$.
- $\langle p,q\rangle = \langle q,p\rangle$.

(b) Show that for P_3, the formula from the previous part does not define an inner product. (*Hint*: Consider the polynomial $p(x) = x(x-1)(x-2)$.)

(c) Now consider the inner product on P_3 by sampling at the points $0,1,2,3$.

$$\langle p,q\rangle = \sum_{i=0}^{3} p(i)q(i)\,.$$

Does this define an inner product on P_3? Do the points where we sample the polynomials matter?

(d) How can we define an inner product on P_n?

(e) Consider the polynomials $p = t-1$, $q = t^3 - 2t^2$ in P_3. Compute their inner product on P_3. Are they orthogonal?

(f) Consider the polynomial $r = t^2$. For any nonzero $u \in P_3$, the projection of r onto $u \in P_3$ is

$$\text{proj}_u r = \frac{\langle r,u\rangle}{\langle u,u\rangle} u\,.$$

Compute $\text{proj}_p r$. (From (e), $p = t-1$)

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