1. The DFT basis as a polynomial basis

Earlier in 16, we viewed the Discrete Fourier Transform (DFT) as a sequence of projections of a length \( n \) signal \( \vec{x} \) onto the set of \( n \) sampled complex sinusoids generated by the \( n \)-th roots of unity. The \( j \)th DFT basis vector represents a (normalized) clockwise path through the \( n \)-th roots of unity starting at 1 and then moving by \( j \) at a time. So the 0th DFT basis vector just stays at 1. The 1st DFT basis vector goes through the \( n \) roots of unity one at a time clockwise starting with 1. The 2nd DFT basis vector takes two trips around the unit circle starting at 1 and moving two at a time.

In this discussion, we will explore an alternative interpretation of the DFT basis vectors. The \( j \)th DFT basis vector will be viewed as \( \frac{1}{\sqrt{n}} \) and represented as an \( n \)-length column by evaluating the expression with \( t \) being the \( n \)th roots of unity. This alternative interpretation allows us to think about interpolation more easily.

Recap of DFT: We can think of a real-world signal that is a function of time \( x(t) \). By recording its values at regular intervals, we can represent it as a vector of discrete samples \( \vec{x} \), of length \( n \).

\[
\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[n-1] \end{bmatrix}
\]

Let \( \vec{X} = [X[0] \ldots X[n-1]]^T \) be the signal \( \vec{x} \) represented in the frequency domain, that is

\[
\vec{X} = U^{-1}\vec{x} = U^*\vec{x}
\]

where \( U \) is a matrix of the DFT basis vectors (\( \omega = e^{i\frac{2\pi}{n}} \)).

\[
U = \begin{bmatrix} \vec{u}_0 & \cdots & \vec{u}_{n-1} \\ \vec{u}_0 & \cdots & \vec{u}_{n-1} \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}
\]

Alternatively, we have that \( \vec{x} = U\vec{X} \) or more explicitly

\[
\vec{x} = X[0]\vec{u}_0 + \cdots + X[n-1]\vec{u}_{n-1}
\]

In other words, \( \vec{x} \) is a linear combination of the complex exponentials \( \vec{u}_i \) with coefficients \( X[i] \).

Let \( s_j = \omega^j \) be the \( j \)th \( n \)th-root-of-unity. The \( n \) points \( s_0, s_1, \ldots, s_{n-1} \) are the \( n \)th-roots-of-unity.

(a) For \( n = 3 \), please sketch the \( s_j \) on the complex plane.

**Answer:** The roots of \( z^3 = 1 \) are \( e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}} \) and \( e^{i\frac{2\pi}{3}} \). On the complex plane, they are 1 + 0i, \( \frac{-1}{2} + \frac{\sqrt{3}}{2}i \) and \( \frac{-1}{2} - \frac{\sqrt{3}}{2}i \), which are uniformly distributed on the unit circle.
(b) Create 3-vectors \( \tilde{b}_j \) for \( j = 0, 1, 2 \) by stacking up the evaluations of \( t^j \) over the \( s_i \). The \( i \)-th entry in \( \tilde{b}_j \) should be \( s_i^j \).

**Answer:**
\[
\tilde{b}_0 = \begin{bmatrix}
(e^{i\frac{2\pi}{3} 0})^0 \\
(e^{i\frac{2\pi}{3} 1})^0 \\
(e^{i\frac{2\pi}{3} 2})^0
\end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \quad \tilde{b}_1 = \begin{bmatrix}
(e^{i\frac{2\pi}{3} 0})^1 \\
(e^{i\frac{2\pi}{3} 1})^1 \\
(e^{i\frac{2\pi}{3} 2})^1
\end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2}i \\
\frac{1}{2} - \frac{\sqrt{3}}{2}i \\
\frac{1}{2} + \frac{\sqrt{3}}{2}i
\end{bmatrix} \quad \tilde{b}_2 = \begin{bmatrix}
(e^{i\frac{2\pi}{3} 0})^2 \\
(e^{i\frac{2\pi}{3} 1})^2 \\
(e^{i\frac{2\pi}{3} 2})^2
\end{bmatrix} = \begin{bmatrix} \frac{1}{4} - \frac{3}{4}i \\
\frac{1}{4} + \frac{3}{4}i \\
\frac{1}{4} - \frac{3}{4}i
\end{bmatrix}
\]

(c) How are these vectors \( \tilde{b}_j \) related to the DFT basis for \( n = 3 \)?

**Answer:** They are scaled column vectors of the DFT basis for \( n = 3 \). \( \tilde{u}_0 = \frac{1}{\sqrt{3}} \tilde{b}_0 \), \( \tilde{u}_1 = \frac{1}{\sqrt{3}} \tilde{b}_1 \), \( \tilde{u}_2 = \frac{1}{\sqrt{3}} \tilde{b}_2 \).

(d) Recall that the DFT matrix can be represented as

\[
U = \begin{bmatrix}
\tilde{u}_0 & \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_{-2} & \tilde{u}_{-1}
\end{bmatrix}
\]

Connect this interpretation to the \( t^j \).

**Answer:** Here we need to show that \( \tilde{u}_{-k} = \tilde{u}_{n-k} \). The vector \( \tilde{u}_{n-k} \) is based on the \( n - k \)-th \( n \)-th-root-of-unity, \( e^{i\frac{2\pi}{n} (n-k)} \). The \( m \)-th entry of \( \tilde{u}_{n-k} \) is \( \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n} (n-k)m} = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n} (mn + \frac{2\pi}{n} (k-m))} = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n} e^{i\frac{2\pi}{n} (k-m)}}, \) because \( m \) is an integer, \( \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n} e^{i\frac{2\pi}{n} (k-m)}} = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n} (n-k)(m)} \), which is the \( m \)-th entry of \( \tilde{u}_{-k} \).

(e) Compute the DFT coefficients \( \tilde{X} \) for the following signal:

\[
\tilde{x} = \begin{bmatrix}
\sin\left(\frac{2\pi}{3} \cdot 0\right) & \sin\left(\frac{2\pi}{3} \cdot 1\right) & \sin\left(\frac{2\pi}{3} \cdot 2\right) & \sin\left(\frac{2\pi}{3} \cdot 3\right) & \sin\left(\frac{2\pi}{3} \cdot 4\right) & \sin\left(\frac{2\pi}{3} \cdot 5\right)
\end{bmatrix}^T.
\]

**Answer:** Recall Euler’s formula: \( e^{ix} = \cos(x) + i\sin(x) \). Hence \( \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \), \( \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \).

Here the \( m \)-th entry of the signal is \( \sin\left(\frac{2\pi}{3} \cdot m\right) \) = \( e^{i\frac{2\pi}{3} (2m)} - e^{-i\frac{2\pi}{3} (2m)} \) = \( \sqrt{6}(\tilde{u}_2[m] - \tilde{u}_{-2}[m]) \). Therefore, \( X[2] = \frac{\sqrt{6}}{2} \), \( X[4] = -\frac{\sqrt{6}}{2} \), while others are zero.

(f) Now, consider real six-dimensional vectors. Suppose we knew

\[
\tilde{x} = \begin{bmatrix}
1 \\
x[1] \\
x[2] \\
1 \\
1 - \frac{\sqrt{3}}{2} \\
x[5]
\end{bmatrix},
\]

where the \( x[i] \) variables indicate values that we do not know. Suppose we further knew that in the frequency domain that \( X[m] = 0 \) for \( |m| \geq 2 \).

Find the missing values that we do not know. What is \( \tilde{X} \)?

**Answer:** We only need to find three variables, \( X[0] \), \( X[1] \) and \( X[-1] \), and we can use \( \tilde{x} = U \tilde{X} \) to write out three equations:

\[
1 = x[0] = \frac{1}{\sqrt{6}} (X[0] \omega^0 \cdot 0 + X[1] \omega^0 \cdot 1 + X[2] \omega^0 \cdot 2 + X[3] \omega^0 \cdot 3 + X[-2] \omega^0 \cdot (-2) + X[-1] \omega^0 \cdot (-1))
\]

\[
1 = x[3] = \frac{1}{\sqrt{6}} (X[0] \omega^3 \cdot 0 + X[1] \omega^3 \cdot 1 + X[2] \omega^3 \cdot 2 + X[3] \omega^3 \cdot 3 + X[-2] \omega^3 \cdot (-2) + X[-1] \omega^3 \cdot (-1))
\]

\[
1 - \frac{\sqrt{3}}{2} = x[4] = \frac{1}{\sqrt{6}} (X[0] \omega^4 \cdot 0 + X[1] \omega^4 \cdot 1 + X[2] \omega^4 \cdot 2 + X[3] \omega^4 \cdot 3 + X[-2] \omega^4 \cdot (-2) + X[-1] \omega^4 \cdot (-1))
\]
We can get $X[0] = \sqrt{6}$, $X[1] = \frac{\sqrt{6}}{2}$ and $X[-1] = -\frac{\sqrt{6}}{2}$.

\[
\begin{align*}
\vec{x} &= \begin{bmatrix}
1 + \sin \left( \frac{2\pi}{6} (0) \right) \\
1 + \sin \left( \frac{2\pi}{6} (1) \right) \\
1 + \sin \left( \frac{2\pi}{6} (2) \right) \\
1 + \sin \left( \frac{2\pi}{6} (3) \right) \\
1 + \sin \left( \frac{2\pi}{6} (4) \right) \\
1 + \sin \left( \frac{2\pi}{6} (5) \right)
\end{bmatrix}.
\end{align*}
\]

(g) What if we didn’t know that $x[4]$ is $1 - \frac{\sqrt{3}}{2}$? Would there be a unique $\vec{x}$ that is compatible with the given information?

**Answer:** If we don’t know the value of $x[4]$, we can only write out two equations to solve three variables. Hence $\vec{x}$ is not unique.

Now consider

\[
\vec{y} = \begin{bmatrix}
y[1] \\
y[2] \\
1 \\
1
\end{bmatrix}.
\]

(h) Suppose instead that we knew that $Y[-2] = 0$ and $Y[-1] = 0$. Find the missing values that we do not know. What is $\vec{Y}$?

(i) Suppose instead that we knew that $Y[0] = 0$ and $Y[1] = 0$. Find the missing values that we do not know. What is $\vec{Y}$?

(j) Would the same approach we used above still work if we instead knew that $Y[-1] = 0$ and $Y[2] = 0$? Why or why not?

**Answer:** For the three questions above, we can do the similar thing to write down three equations with three variables, but $\vec{y}$ will be a complex vector.

(k) Consider a length $n$ discrete-time signal $\vec{x}$, along with its DFT coefficients, $\vec{X}$. If we know $X[m] = 0$, for all $|m| > k$, what is the minimum number of sampling points we need to interpolating the $\vec{x}$?

**Answer:** $2k + 1$

(l) (Optional) Given a continues time sinusoidal signal $x(t) = \sin \left( \frac{2\pi}{3} t \right)$, what is its frequency? What is the sampling rate for creating the discrete signal $\vec{x}$ in (e)?

**Answer:** The frequency is $1/3$, so our sampling rate should be higher than $2/3$. The sampling rate for (e) is $1$ Hz.

(m) (Optional) Sample $x(t) = \sin \left( \frac{2\pi}{3} t \right)$ with the sample rate $= 2$Hz between $0 \leq t < 3$. How many data points do you get? Collect those sample points as a discrete signal $\vec{y}$. Compare the DFT coefficients of $\vec{y}$ with the result in (e). Explain their relationship.

**Answer:** 6 samples:

\[
\vec{y} = \begin{bmatrix}
\sin \left( \frac{2\pi}{6} (0) \right) & \sin \left( \frac{2\pi}{6} (1) \right) & \sin \left( \frac{2\pi}{6} (2) \right) & \sin \left( \frac{2\pi}{6} (3) \right) & \sin \left( \frac{2\pi}{6} (4) \right) & \sin \left( \frac{2\pi}{6} (5) \right)
\end{bmatrix}^T.
\]
(n) (Optional) Sample \( x(t) = \sin(\frac{2\pi}{3}t) \) with the sample rate = 2Hz between \( 0 \leq t < 6 \). How many data points do you get? Collect those sample points as a discrete signal \( \vec{z} \). Compare the DFT coefficients of \( \vec{z} \) with the result in (e). Explain their relationship.

**Answer:** \( \vec{z} \) has 12 points.

(o) (Optional) Sample \( x(t) = \sin(\frac{2\pi}{3}t) \) with the sample rate = 2/3 Hz between \( 0 \leq t < 3 \). Are you able to reconstruct the signal based on the sample points?

**Answer:** No